Senior Challenge ’06. Solutions

1. Prime starter

Suppose $p = x^2 - y^2 = (x + y)(x - y)$. Since p is prime then $x - y = 1$, $y = x - 1$, $p = 2x - 1$ and $x = (p + 1)/2$, $y = (p - 1)/2$.

2. Uncommon children

No. Counterexample:

Jon Smith Yr 7, Mary Smith, Yr 8, Jon Green, Yr 8, Mary Green, Yr 7

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3. Trapped square

The area of the triangle is 6, the triangle being right-angled There are two possible squares to consider. The most obvious one shares a right angle with the right angle of the triangle. The equation that you have to solve for the side length x of the square is $(3 - x)/x = x/(4-x)$, with the unique solution $x = 12/7$. So the area of that square is 144/49. But one should also consider the largest square inside the triangle one of whose sides lies along the hypotenuse of the triangle. A slightly more complicated argument shows that its side-length is 60/37, slightly less than 12/7.

4. Bacterial warfare

It is impossible. There is a total of 21 million bacteria, and whenever a membrane is removed an even number of millions of bacteria die. Since 21 is an odd number, the total number of bacteria reduces in this way always to an odd multiple of a million, never zero.

5 Integral triangle

(i) The simplest solution is to paste together two copies of the 3, 4, 5 triangle, one turned over and the two joined along the side of length 4. The resulting triangle has sides of length 5, 5 and 6, and has area 12.

(ii) The word ‘integral’ is an anagram of the word ‘triangle’.

6 Challenging hexagon

Each of the four triangles in the diagram has the same area. This is because the sines of two angles whose sum is 180^o are equal. Such angles are known as ‘complementary angles’. The total area is then the sum of the squares of the three sides plus four times the area of any one of the triangles,