SENIOR CHALLENGE '04

Sponsored by Mercer Human Resource Consulting Ltd

With Answers

1. This competition is open to all pupils at Schools in and around Merseyside who are under the age of fifteen and a half (born on or after 1 September, 1988). It is to be tackled at home, during half-term.

2. Your entry must be your own unaided effort, though of course you may refer to books, etc., for ideas on how to start, and may ask the meanings of unfamiliar words. *More marks will be given if you explain clearly how you get your answers.*

3. We hope that you enjoy the questions. It is possible to win a prize even though you may not have attempted all the questions, so do let us have your entry even if it is not quite finished!

4. Hand your neatly written entry, with your name on every page, to your class teacher as soon as possible after half-term.

5. Prizes for overall winners and many consolation prizes will be presented at an Evening of Mathematical Recreation at the University of Liverpool in May. Certificates will be awarded to all who do well.

6. Solutions will be posted on www.maths.liv.ac.uk/∼mem/ early in March.

This competition is promoted by Mathematical Education on Merseyside (MEM).
(Registered Charity No 517028.)
The Department of Mathematical Sciences, The University of Liverpool, Liverpool, L69 7ZL.
The drawings are by Peter H. Ackerley.

MEM is pleased to acknowledge generous financial help received from the following during the past year:

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1. WINNING NUMBER

There was a sixteen horse race in progress at the racecourse, the horses being numbered 1 to 16, but I missed the finish. I asked six of my friends to tell me the number of the winner. These were their answers.

a) It was even

b) It was odd

c) It was prime

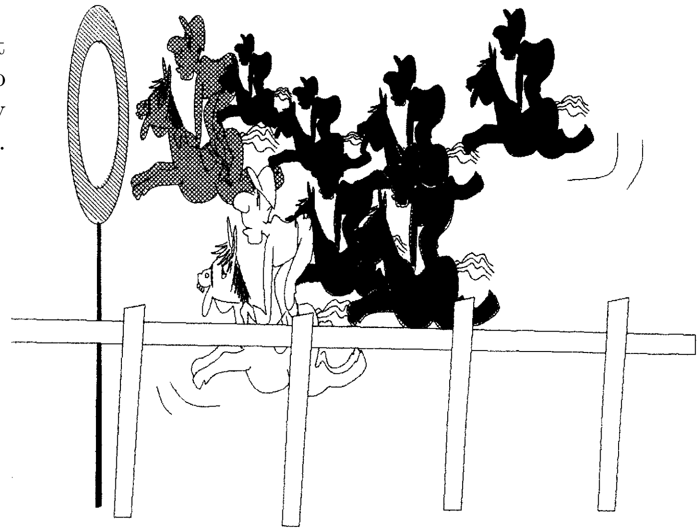
d) It was a square number

e) It had two digits

f) It was between 6 and 12

However, only four had told the truth.

Which number was the winner?



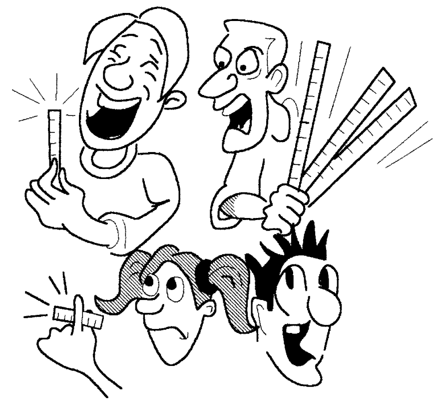
Answer: The winning number was 11.

2. COFFEE BREAK

A coffee pot holds four times as much as a milk jug. Sally and her friends all have the same size cup of coffee but they like different amounts of milk in their coffee. Sandeep likes half coffee and half milk; Sharma, Suraj and Sophie each like one quarter milk, and Sally likes her coffee black. Sally fills the five cups in the way each person likes, and just one of the containers is emptied. How many cupfuls of liquid are left in the other?

Answer There were $1\frac{1}{4}$ cupfuls of coffee left in the coffee pot.

3. RULERS NOT OK



There were 19 students in a class when a teacher brought in a box containing rulers. The first student was given one nineteenth of the rulers plus one nineteenth of a ruler, the second student was given one eighteenth of the remainder plus one eighteenth of a ruler and so on until there were only two students remaining. The last but one student took one half of the remainder and half of a ruler. The last student felt a little aggrieved.

a) Why did she feel aggrieved?

b) How many rulers were there in the box to start with?

Answer The cartoon was maybe slightly misleading. The rulers would have been no use to anyone if they had been broken. In fact there were just 18 rulers, and each student got one complete ruler, except the last who did not get a ruler at all and so naturally felt aggrieved.

4. SQUARE THIS

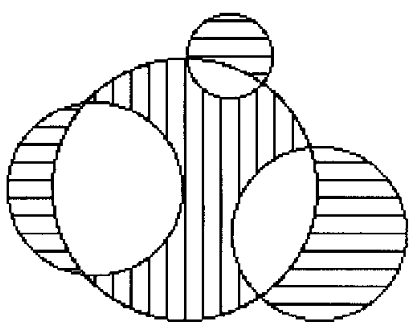
Is it possible to find three whole numbers a, b, c, none of which is zero or a perfect square, such that

$$\sqrt{a} + \sqrt{b} = \sqrt{c} \ \ ?$$

Solution If you square the equation you get

$$a + 2\sqrt{(ab)} + b = c.$$

So all you need to do is to choose a and b so that neither is a square, but ab is. For example, choose $a = b = 2$, when $c = 8$, or $a = 2, b = 8$, when $c = 18$.

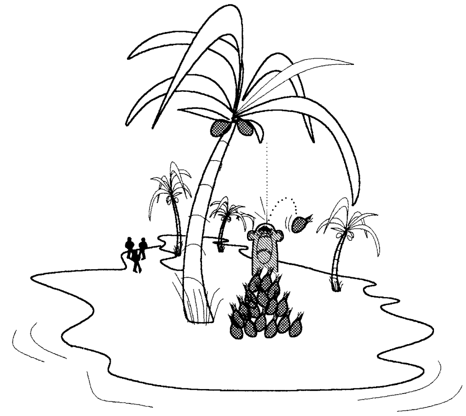


5. SHADY BUSINESS

The largest of these circles has diameter 6 units, the smallest, 2 units, and the other two 4 units each. Which is the larger, the vertically shaded area or the horizontally shaded areas taken together?

Solution The area of the large circle is 9π and the areas of the smaller circles are $4\pi, 4\pi$ and π, which sum to 9π. Let x be the total area of the three white parts of the circles. Then the areas of the shaded regions are both equal to $9\pi - x$.

6. COCONUT MATTERING



Three men crash-land their airplane on a deserted island in the South Pacific. On their first day they gather as many coconuts as they can find into one big pile. They decide that, since it is getting dark, they will wait until the next day to divide the coconuts.

That night each man took a turn watching for rescue searchers while the others slept. The first watcher got bored so he decided to divide the coconuts into three equal piles. When he had done this, he found that he had one remaining coconut. He gave this coconut to a monkey, took one of the piles, and hid it for himself. Then he jumbled up the two other piles into one big pile again.

To cut a long story short, each of the three men ended up doing exactly the same thing. They divided the coconuts into three equal piles and had one extra coconut left over, which they gave to the monkey. They each took one of the three piles and hid those coconuts. They each came back and jumbled up the remaining two piles into one big pile. What is the smallest number of coconuts there could have been in the original pile?

Answer The smallest number of coconuts there could have been was 25. After the first airman took his 8, with 1 to the monkey, there were 16. After the second took 5, with 1 to the monkey, there were 10. Finally, after the third took 3. with 1 to the monkey, there were just 6 left. When they divided these equally in the morning the monkey got none. No smaller number of the form $3k + 1$ works.

7. IMPOSSIBLE QUESTION

Many whole numbers can be written as the difference of two perfect squares. For example:

$$5 = 9 - 4, \qquad 20 = 36 - 16, \qquad 21 = 25 - 4.$$

Which cannot be written as the difference of two perfect squares (one of which may be 0)?

Solution We note first that $a^2 - b^2 = (a + b)(a - b)$. There are then two cases. If one of a and b is even and the other odd then $a^2 - b^2$ is odd, $= 2k + 1$, say. Moreover, for any k, $2k + 1 = (k + 1)^2 - k^2$. On the other hand if a and b are both even or both odd then $a^2 - b^2$ is a multiple of 4, $4k$, say, and for any k, $4k = (k + 1)^2 - (k - 1)^2$. So the numbers that cannot be expressed as the difference of two squares are exactly those of the form $4k + 2$.