Solutions to Senior Challenge 03

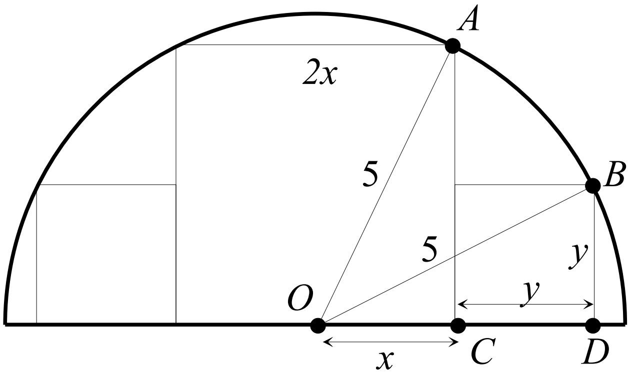
1. **Cuisenaire** The largest possible length is $8 + 4 + 2 + 1 = 15$ and experiment shows that all smaller lengths can in fact be made with the given rods. So the answer is 16. [In fact this is just the same as writing a number in base 2: for example $11_{10} = 1011_2$, that is 11 in base 10 is written 1011 in base 2, indicating that rods 8, 2, 1 are needed. More generally, lengths of $1, 2, 4, \dots 2^n$ make up all lengths $< 2^{n+1}$.]

2. **Unexpected Guest** Suppose that a fraction x of each piece is cut off; this represents a fraction $\frac{x}{6}$ of the whole pizza. Putting the six cut-off pieces together makes a fraction $6 \times \frac{x}{6} = x$ of the whole pizza. But we want this to equal $\frac{1}{7}$ so that all seven guests receive one-seventh of the pizza. So $x = \frac{1}{7}$.

3. **Good Dogs** When Fido is finished with a piece of carpet the number of pieces has increased by *three* (one piece becomes four) and when Trusty is finished with a piece the number has increased by *six* (one piece becomes seven). So however many times either dog does his or her thing, the number of pieces at the end will have the form 1 + a multiple of 3. Now $2003 = 667 \times 3 + 2$, that is *two* more than a multiple of 3. So 2003 cannot be the full tally of carpet pieces: there must be at least two more hidden away somewhere.

   Note that this depends on the special numbers 4 and 7, each one more than a multiple of 3. If these are replaced by say 4 and 6 then we are asking which whole numbers can be written in the special form $1 + 3n + 5m$ for whole numbers n and m. This is also an interesting problem, and in fact every number ≥ 9 can be written in this way, so any final number of carpet pieces from 9 upwards is possible.

4. **Gardener's Question Time** This is tricky but can be done entirely with Mrs Pythagoras's husband's theorem about right-angled triangles. (In fact legend has it that it was exactly to solve this problem that Mr Pythagoras invented his theorem.) The radius of the circle is 5, and using the notation of the figure $OA = OB = 5$.



   Let the large square have side $2x$. By symmetry the centre of the circle must be at the midpoint of the bottom side of the larger square for this square to fit neatly inside the semicircle. Using (Mr) Pythagoras's theorem on triangle OAC gives $x^2 + (2x)^2 = 5^2$, which gives $5x^2 = 25$, that is $x = \sqrt{5}$. Using the same theorem on triangle OBD we get $(x + y)^2 + y^2 = 5^2$, which gives $x^2 + 2xy + 2y^2 = 25$ and on using $x = \sqrt{5}$ we get $y^2 + y\sqrt{5} - 10 = 0$. This can be written $(y - \sqrt{5})(y + 2\sqrt{5}) = 0$ and the only positive solution to this is $y = \sqrt{5}$. So in fact the small square has exactly half the side length of the large square. The total area of the squares is $(2x)^2 + 2y^2 = 30$, using $x^2 = y^2 = 5$. The total area of the semicircle is $\frac{1}{2}\pi \times 5^2 = 39.27$ approximately. So the area left for planting is about 9.27 square metres.

5. **Doing The Splits** Let n and m be the two three-figure numbers. Putting n to the left of m makes the number $1000n + m$. So we are trying to find solutions of

$$1000n + m = 7nm. \qquad (1)$$

The hint is intended to suggest that you try $n = m$ since the left side is then $1001n$, making the equation $1001n = 7n^2$ or $1001 = 7n$. As 1001 is divisible by 7 this leads to $n = m = 143$.

But to get bonus marks you should check that there are no other solutions. We are looking for 3-figure numbers so n and m lie between 100 and 999. Now (1) can be rewritten as $m = n(7m - 1000)$, showing that m is an exact multiple of n, but because of the restriction on the size of m and n it also shows that $7m - 1000$ must be one of the numbers $1, 2, 3, \ldots, 9$. We have only to examine these values one by one. Taking $7m - 1000 = 1$ gives the answer already obtained, $m = n = 143$. The other values apart from 8 do not give whole numbers for m, and 8 gives the slightly phony answer $m = \frac{1008}{7} = 144$: phony because it leads to $n = 18$, not a three-figure number. This is the 'solution' $018144 = 7 \times 18 \times 144$.

For the second part we are trying to solve

$$1000n + m = nm.$$

As before rewriting this gives $m = n(m - 1000)$. But $m \leq 999$ so the right hand side is negative, showing this to be an impossibility. Of course you can analyse other cases such as $1000n + m = 2nm$ in the same way.

6. **Running Mates** We need to use the formula 'time taken = distance travelled divided by speed'. Let d be the distance between the houses, let c be Chris's speed and let a be Alex's speed. When they first meet,

$$\frac{d - 600}{a} = \frac{600}{c}, \text{ that is } \frac{a}{c} = \frac{d - 600}{600}.$$

When they meet for the second time,

$$\frac{2d - 400}{a} = \frac{d + 400}{c}, \text{ that is } \frac{a}{c} = \frac{2d - 400}{d + 400}.$$

We now have two expressions for $\frac{a}{c}$ and equating them gives

$$\frac{d - 600}{600} = \frac{2d - 400}{d + 400}, \text{ that is } d^2 - 1400d = 0$$

after cross-multiplying and simplifying. The only solution apart from $d = 0$ (which hardly seems to be practical!) is $d = 1400$ metres.

Notice that we cannot find the actual speeds of the runners from this, only the ratio of their speeds, $\frac{a}{c}$, which comes to $\frac{4}{3}$.

7. **Square Bashing** The key to this is really symmetry. With a board which has both dimensions even or both odd, the first player should paint a square whose centre is at the centre of the board. After that, every move by the second player should be answered by the first player by painting a square which is symmetrically placed to the second player's, by a rotation through 180° about the centre of the board. The first player is always assured of a legal move and so must finish painting the board.