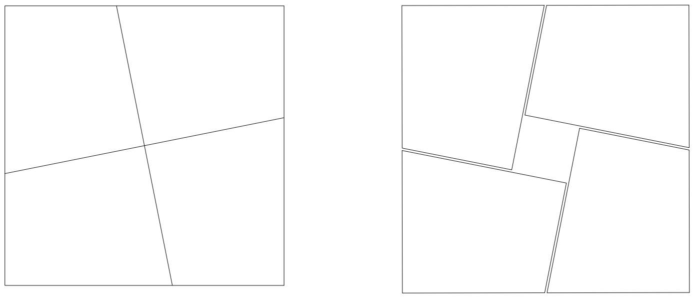
Solutions to Senior Challenge 02

1. **Squares**

   From the diagram, the hole is a 1×1 square.



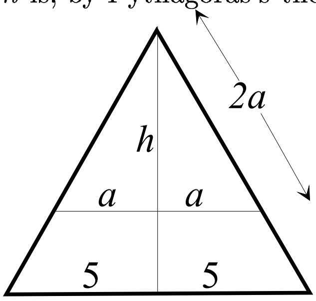
2. **A Russian Tale**

   Let A, B, C be the numbers of mushrooms collected by Anton, Boris and Carla respectively. Then A is 80% of B, i.e. $A = \frac{4}{5}B$, making $B = \frac{5}{4}A = 375$. Similarly C is 120% of B, that is $C = \frac{6}{5}B$, so $C = 450$.

3. **Helipad**

   Here is a solution which avoids the use of trigonometry.

   The key observation is that if we draw a line across the triangle so as to cut off a length of $2a$, as shown, then the line cuts off another equilateral triangle (of side $2a$), so the height h is, by Pythagoras's theorem, $h = a\sqrt{3}$, and the area above the line is $a \times a\sqrt{3} = a^2\sqrt{3}$.



   In particular, of course, the area of the whole triangle is $25\sqrt{3}$ sq.m. One-third of this is $25\sqrt{3}/3$ and we want to draw the first line so that the area above it equals this amount, so we want $a^2\sqrt{3} = 25\sqrt{3}/3$, that is $a = 5/\sqrt{3}$. The height h of this triangle is $a\sqrt{3} = 5$, that is we need to draw the first line a distance of 5 below the apex of the triangle.

   For the second line, of length $2b$, say, we want $b^2\sqrt{3}$ equal to *two-thirds* of the total area, that is $50\sqrt{3}/3$. This gives $b = \sqrt{(50/3)}$ and the corresponding height is $b\sqrt{3} = \sqrt{50}$: the second line needs to be drawn at a distance below the apex of $\sqrt{50} = 7.07$ to 2 decimal places. (The whole height of the helipad triangle is $5\sqrt{3} = 8.66$ to 2 decimal places.)

4. **The Party's Over**

Here is an algebraic solution; of course intelligent trial and error will also produce the answer. Let there be m married couples altogether, and s single people. If everyone said goodbye to everyone else then all the $2m + s$ people would say goodbye to $2m + s - 1$ others, and there would be $(2m + s)(2m + s - 1)$ goodbyes said. But we have to subtract the $2m$ goodbyes which are not said by husbands to wives and vice versa. So the total is

$$4m^2 - 4m + 4sm + s^2 - s = 102.$$

(Notice incidentally that this implies $4m^2 - 4m < 102$ and so $m(m-1) < 25$ and $m \leq 5$. But in practice we would probably take $m = 0, 1, 2, 3...$ in succession and use the equation above to try to find s.) Each m actually gives a quadratic equation for s, and if the formula for solving a quadratic is not known, then trial will quickly decide whether there is a whole number solution. For example with $m = 1$ (so no married guests) we get $3s + s^2 = 102$ which by trying $s = 1, 2, 3\ldots$ quickly shows that no whole number solution exists. In fact $m = 4$ is the first value which gives a solution, with the corresponding value of s equal to 3. So there were $m - 1 = 3$ married couples besides the hosts at the party.

5.

Now For The Washing Up

Of course, Gerald ended with the same number that he began with. Here some algebra makes the process much easier to explain; otherwise all one can do is to look at many cases.

Let the original figures be a, b so that the actual value of the number chosen by Gerald is $10a + b$. Then the number obtained by subtracting the digits from 9 is $10(9-a) + (9-b) = 99 - 10a - b$.

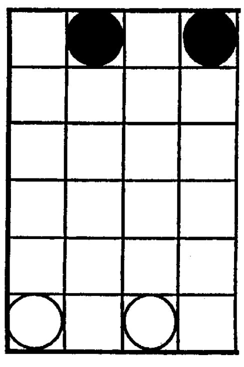
Next, the two numbers are put together. This is the same as multiplying the first number by 100 and adding the second one:

$$100(10a + b) + 99 - 10a - b = 990a + 99 + 99b.$$

Dividing by 11 gives $90a + 9 + 9b$: notice this shows that the result will always be a whole number.

Subtacting 9 gives $90a + 9b$ and dividing by 9 gives $10a + b$, which is what Gerald started with. Hey presto!

6. **CAN SHE DO IT?**



Yes, Lou can, by simply moving her left-hand counter steadily forwards, that is *downwards*, leaving for later the other one, which could not be taken off anyway by Mike, by the rules of the game. It is only after every two moves that Mike can have his counters in a position to capture Lou's. The danger moment is after Lou's third move, when the best that Mike can do is to guard her next square. But he needs two moves to shift his attack to the square that Lou is on, by which time she has slipped through.

7. **PENTAJIG**

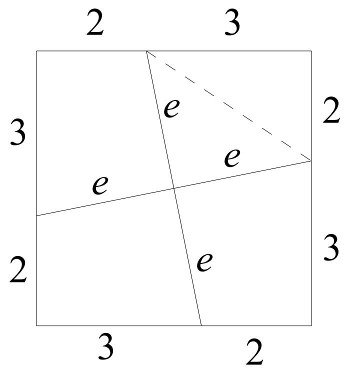
There are several solutions to this one. We have used numbers for the five colours. With the first row as shown here there is only one place for 1 in the second row, and only two possible places for 1 in the third row. Further choices have led to the two solutions shown.

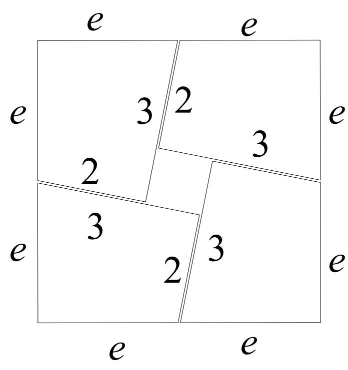
1	2	3	4	5
5	4	2	1	3
3	5	1	2	4
2	3	4	5	1
4	1	5	3	2

1	2	3	4	5
3	5	2	1	4
4	3	5	2	1
5	4	1	3	2
2	1	4	5	3

8. **Rectangles**

In this figure, the larger square has side $2e$ and using Pythagoras's theorem on two triangles which have the dashed lines as hypotenuse, we get

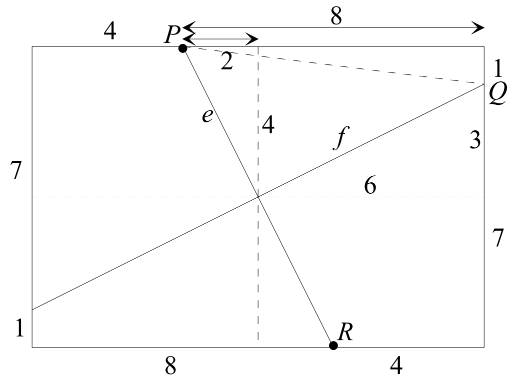


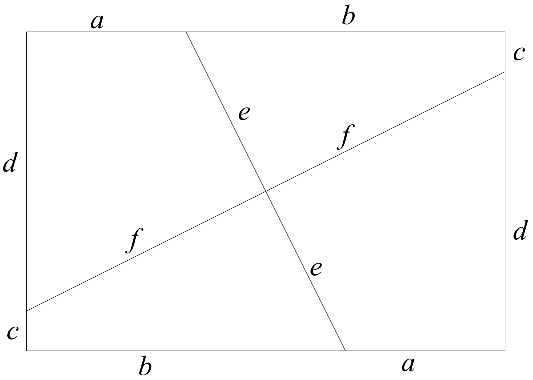


$e^2 + e^2 = 3^2 + 2^2$ so $e^2 = 6.5$ which makes $2e = 5.099$ to 3 decimal places. So you can see this square is only a tiny bit bigger than the original one. (You could also say that, using areas, $(2e)^2 - 1 = 25$, as the hole has area 1 and the pieces making up the original square, of area 25, have only been rearranged. This gives $(2e)^2 = 26$, giving the same answer for $2e$.)

In the figure, dashed lines have been drawn through the centre of the rectangle, parallel to the sides of the rectangle, and the lengths 2, 3, 4, 6 marked come from simple subtraction of lengths. It follows that $e^2 = 20$ and $f^2 = 45$ by Pythagoras's theorem. Thus $e^2 + f^2 = 65$.

But the square of the length PQ is, also by Pythagoras's theorem, $8^2 + 1^2 = 65$. Finally using Pythagoras's theorem yet again, it follows that the angle at which the sides e, f meet is a right angle. (Another way of seeing this is to show that $PQ = QR$ using Pythagoras's theorem and then to use the fact that the centre of the rectangle is the midpoint of PR.)



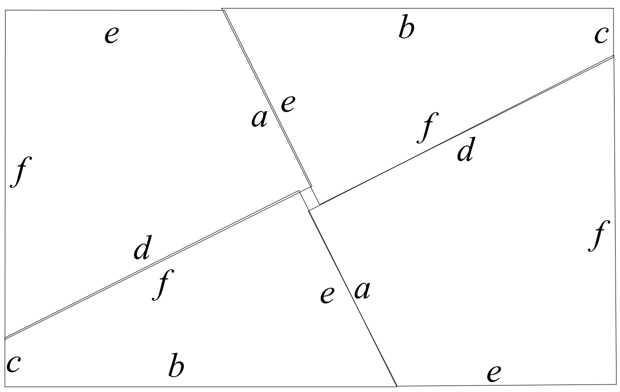


In the second figure, the lengths have been replaced by general lengths a, b, c and d, and the condition that the lines meet at right-angles works out as $a^2 + d^2 = b^2 + c^2$

For finding the different ways of reassembling the pieces, a paper or cardboard model is much recommended! In the pictures below, we use the general letters a, b, c, d; for the problem given, substitute $a = 4, b = 8, c = 1, d = 7$, in which case $e = 2\sqrt{5} = 4.47..., f = 3\sqrt{5} = 6.71...$ using a similar argument to the first one given above for the square. Notice, by the way, that $e/f = (c + d)/(a + b)$ in the special case. (Actually, this always holds, and is not too hard to prove using the fact that each of the four pieces is bounded by a cyclic quadrilateral!)

There are three cases: we can turn over the pieces of sides a, e, f, d; we can turn over the other two pieces, or we can avoid turning any pieces over. (If we turn them all over then this doesn't give anything different.)

The figure shows the case where the pieces of sides a, e, f, d are turned over.



The rectangular hole is here $e - a$ by $d - f$, which is the special case is 0.47 by 0.29 approximately, or exactly $2\sqrt{5} - 4$ by $7 - 3\sqrt{5}$.

When the pieces with sides b, c, f, e are turned over we get a rectangle in the middle with sides $b - e$ and $f - c$, which in the special case is 3.53 by 3.71 approximately, or $8 - 2\sqrt{5}$ by $3\sqrt{5} - 1$ exactly.

In this case, the shapes of the rectangles are the same:

$$\frac{2\sqrt{5}-4}{7-3\sqrt{5}} = \frac{3\sqrt{5}-1}{8-2\sqrt{5}},$$

as can be verified by cross-multiplication. It is an interesting fact that this always holds: assume that $b > e > a, d > f > c$ as in the example. Then the rectangles formed as above have the same shape, that is

$$\frac{e-a}{d-f} = \frac{f-c}{b-e}.$$

We leave this as a nice exercise!

If we don't turn over any of the pieces then the rectangle in the middle becomes $b-a$ by $d-c$, when $b > a, d > c$ as in the example (where this rectangle is 4 by 6). We always have

$$\frac{b-a}{d-c} = \frac{c+d}{a+b}$$

since cross-multiplying this gives the known fact $b^2 - a^2 = d^2 - c^2$. So this rectangle is always the same shape as the original. This case is illustrated below.