Senior Challenge Solutions 2001

1. The three sides of length b add to 12 m, so $4a = 36$, giving $a = 9$m. The area of each pen is $ab = 36$ sq m.

2. This can be solved by trial and error, or by algebra. Using algebra we get $4a + 3b = 48$ and $ab = 48$. From these we deduce

$$4a + 3\frac{48}{a} = 48a, \text{ giving } 4a^2 + (3 \times 48) = 48a, \text{ i.e. } a^2 - 12a + 36 = 0,$$

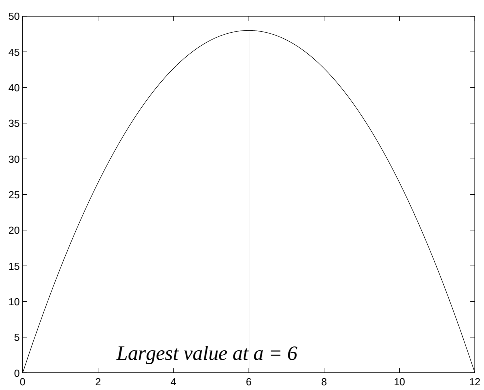
after rearranging and cancelling 4 from both sides. This gives $(a - 6)^2 = 0$ so $a = 6$ is the only solution. The sides are $a = 6,\ b = 48/6 = 8$. Notice that in this case, a is the shorter side.

3. The interesting thing is that all the answers come out to be the same in the bottom line:

Chosen card	1	2	3	4	5	6	7	8	9
Place after one deal and pickup	1	1	1	2	2	2	3	3	3
Place after two deals and pickups	1	1	1	1	1	1	1	1	1

The question doesn't actually specify the order in which the columns besides the one containing the chosen card are picked up, but it's not hard to see this doesn't matter anyway. Notice the simple progression after one deal and pickup.

4. *Note. The solution below assumes that the two pens remain equal rectangles side-by-side as in the original setup. Some entrants took other shapes, such as a circle divided into two semicircles, and due credit was given for all these attemps.*

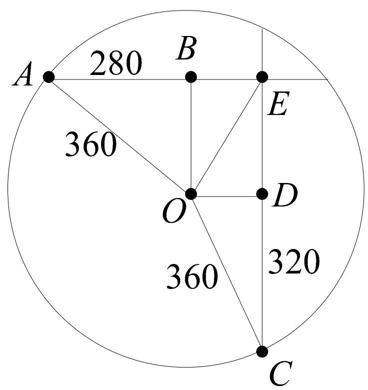


There are several ways to see that Farmer Chris is right. We can use a bit of algebra and a graph, or more algebra. We know that $4a+3b = 48$ and we're interested in how big ab can be. Put $ab = A$ for Area. Then $4a + \frac{3A}{a} = 48$ which gives $A = 16a - \frac{4}{3}a^2$. The question is: how big can $16a - \frac{4}{3}a^2$ be? We could take several values for a and plot a graph, as in the figure, or we can argue without a graph as follows:

$$48 - A = \frac{4}{3}a^2 - 16a + 48 = \frac{4}{3}(a^2 - 12a + 36) = \frac{4}{3}(a - 6)^2 \geq 0,$$

since squares are always ≥ 0. So always $48 - A \geq 0$ so $A \leq 48$ as Chris said. We found in Question 2 that $A = 48$ requires $a = 6$ and $b = 8$. This is a very clever kind of algebraic argument and we mainly expected you to take values and plot a graph or something like that.

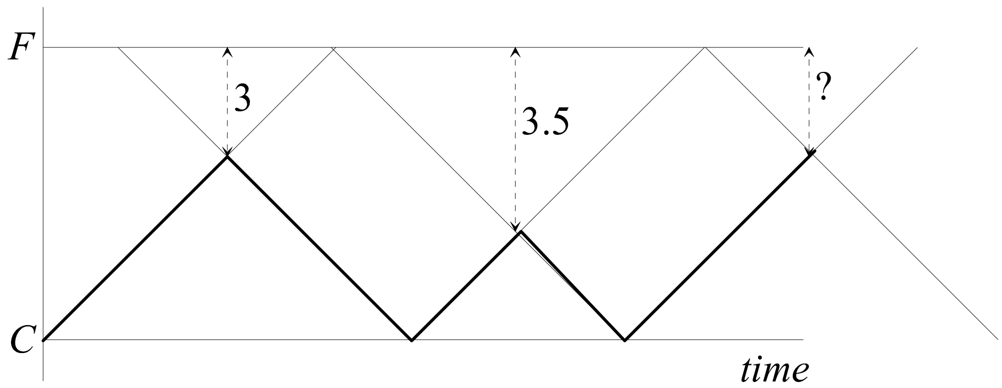
5. At first it seems that the best time is 19 minutes, with A running back and forth taking the others over one by one. But you can do better than that if the two slowest go over *together*. There's no point in doing that unless there is a faster person on the other side who can then come back. The best solution, which can be found by intelligent trial and error, is (indicating the people on the two sides of the bridge by separating them with a vertical bar |):
ABCD | ; CD | AB ; BCD | A ; B | ACD ; AB | CD ; | ABCD
The time used is 2 + 2 + 10 + 1 + 2 = 17 minutes. It doesn't matter whether B comes back first or A, in either case the time is 17 minutes.



6. This question can be done entirely by Pythagoras's theorem (as you would expect since it is Pythagoras who is doing the planting). Referring to the figure, OB and OD are perpendicular to the two strings, and so B and D are the midpoints of the strings. Thus AB is half the length of one string, hence 280 cm, and CD is half the length of the other string, hence 320 cm. Using right-angled triangle ABO we find $OB^2 = 360^2 - 280^2 = 40^2(9^2 - 7^2) = 40^2 \times 32$ and using right-angled triangle CDO we find $OD^2 = 360^2 - 320^2 = 40^2(9^2 - 8^2) = 40^2 \times 17$. But notice also that $OBED$ is a rectangle, so that $BE = OD$. We now get $OE^2 = BE^2 + OB^2 = OD^2 + OB^2 = 40^2(17 + 32) = 40^2 \times 7^2 = 280^2$, so that $OE = 280$ cm.

(In general, if the strings are of length $2a, 2b$, r is the radius of the circle and c is the distance OE, then the same method as above gives $2r^2 = a^2 + b^2 + c^2$. An alternative, slightly harder version of this problem is to give a, b, and c and ask for r. Also notice that all the numbers a, b, c, r in the problem are whole numbers. Maybe you can find other values which make this true (other than just multiplying everything by the same amount). You have to be a little careful, since it's essential to have a and b both less than r.)

7. This question is quite well solved by means of a diagram, as shown. Here C and F stand for Charlie's house and Frankie's house and the vertical axis is distance. The thin line represents Frankie's progress, with time along the horizontal and the scale chosen so that his line is at 45° to the horizontal. The line is *straight* because Frankie moves at a constant speed, so travels the same distance in each unit of time. Likewise Charlie's progress is marked by the bold line, and, because they run at the same speed, his line is also at 45° to the horizontal. You can now see from the diagram the places where they meet and turn round; note that these make straight lines, because Charlies's and Frankie's lines are both 45° lines. You can also see the distances of 3 miles and 3.5 miles. (It is now clear that Charlie starts first, as in the figure.) Also it is clear that the diagram starts to



repeat after Charlie's second visit home, so the answer to the place where they meet for the third time is 3 miles from Frankie's house.

Notice that it's really just the same thing as Charlie running back and forth straight from his house to Frankie's, passing Frankie on the way who is running back and forth from his house to Charlie's. The distance between the two houses is $6\frac{1}{2}$ miles.

8. With a 5×5 array of cards, the position of the card after one deal and pickup (the first row of the table) takes the form
1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5
and the row under that is all 1's: the chosen card always comes to the first place after two deals and pickups.

   In fact with a little thought it becomes clear that what is happening is this. Let there be r rows and c columns. Suppose we know the position p of the chosen card after a particular deal and pickup. To find the position after the *next* deal and pickup, divide p by the number of columns c and take the smallest whole number $\geq \frac{p}{c}$. Let us write this whole number as $\{\frac{p}{c}\}$. For example, with $r = 5$ and $c = 3$, the positions 1, 2, 3, 4, 5, 6, 7, 8, ... at the start become
$\{\frac{1}{3}\} = 1, \{\frac{2}{3}\} = 1, \{\frac{3}{3}\} = 1, \{\frac{4}{3}\} = 2, \{\frac{5}{3}\} = 2, \{\frac{6}{3}\} = 2, \{\frac{7}{3}\} = 3, \{\frac{8}{3}\} = 3$, and so on. For the next deal and pickup we can repeat the process with these new positions 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5 to obtain 1,1,1,1,1,1,1,1,1,2,2,2,2,2,2 and one more deal and pickup will make all the numbers 1: every choice of card at the beginning will come to the first position after at most three deals and pickups.

   So in general, with r rows and c columns, how many deals and pickups does it take for the chosen card to be certain to be in first place? When $r = c$ it's pretty clearly always *two* deals and pickups, as in the 3×3 and 5×5 cases above. In fact this will also be true if $r < c$. The general answer is in fact $\{\frac{r}{c}\} + 1$ deals and pickups to guarantee that the chosen card is in first place.

   You can make a variant of this problem by picking up the column with the chosen card *second* instead of first. In that case is it true that the chosen card will always come to the same position after a suitable number of deals and pickups?