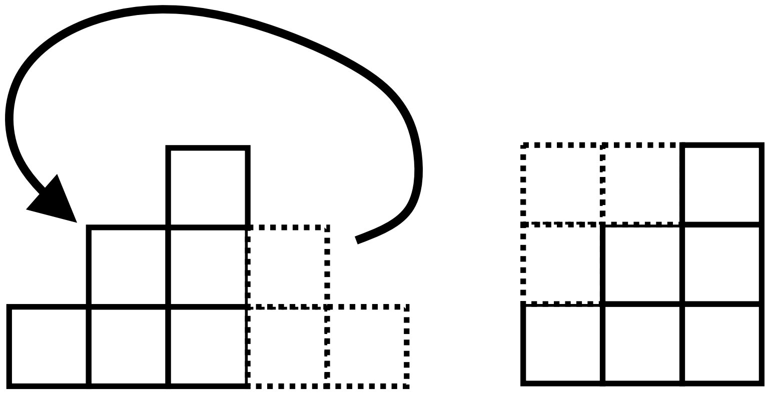
Solutions to Senior Challenge 2000

1. **Cut and Cover**



2. **Spinning Freddy** Let d be the distance Fred rides. Then

$$\frac{1}{3}d + \frac{1}{7}d + 1 = \frac{1}{2}d, \quad \text{i.e. } d\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{7}\right) = 1,$$

giving $d = 42$ miles.

3. **High Powered**

$$2^3 > 7, \quad \text{so} \quad (2^3)^{1000} > 7^{1000}, \quad \text{i.e.} \quad 2^{3000} > 7^{1000}.$$

$$2135 = 5 \times 427; \quad 5978 = 14 \times 427, \quad 7^5 = 16807, \quad 2^{14} = 16384 < 7^5,$$

these being possible by hand or on an ordinary calculator, so

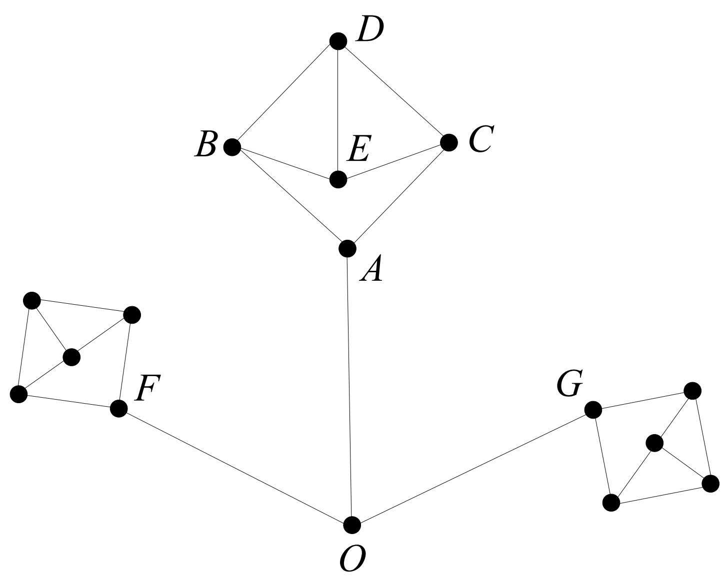
$$7^5 > 2^{14}, \quad \text{giving} \quad 7^{2135} = (7^5)^{427} > (2^{14})^{427} = 2^{5978}.$$

4. **All in Good Time** There will be $100 \times 12 = 1200$ months. There will be $365 \times 100 + 25$ days, counting the 25 leap years 2000, 2004, ..., 2096 as giving an extra day each. This is 36525 days, which is 5217 weeks and 6 days, i.e. 5217 complete weeks. [The same is true of any century, even one that does not begin with a leap year, for the number of days then goes down by 1 making 5217 weeks and 5 days.]

5. **Joan and Jim** It *is* impossible. For counting 3 friends for each of the 11 people, we count each pair of friends *twice*, i.e. $33 = 2\times$ the number of pairs, which is impossible. Clearly the number of party-goers must be *even* for everyone to have exactly 3 friends.

6. **Tall Story** With n layers there are exactly n^2 bricks, using a cut such as that in Question 1 to turn the wall (in our mind's eye!) into a perfect square. There are 5217 weeks (and 6 days) by Question 4, so we want the nearest square to 5217, which is $72^2 = 5184$. So there should be $n = 72$ layers and $2n - 1 = 143$ blocks in the bottom layer. We want to calculate the date of the Saturday which is exactly 5184 weeks after 1 January 2000. Besides these 5814 weeks there are 33 weeks and 6 days = 237 days up to and including 31 December 2099. Counting backwards from 31 December 2099, calling this 'day one', we want 'day 237'. Counting back to 1st May 2099 from 31 December gives 245 days (1 May is 'day 245'), so 'day 237' is 9 May 2099: on this Saturday the last brick will be laid.

7. **Jim and Joan** It *is* impossible. Consider the 5 dots (people) in one of the 'end-groups', such as A, B, C, D, E shown in the figure.



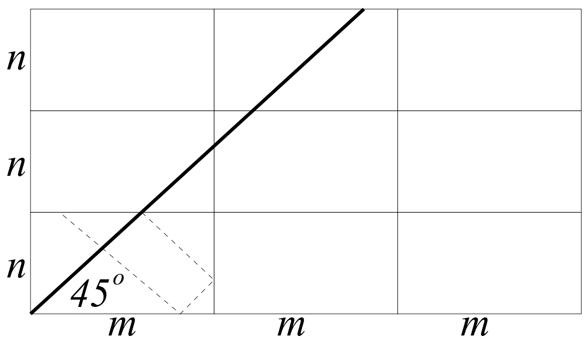
If A is paired to B or C then the remaining three among A, B, C, D, E cannot be paired. So A must be paired to the central dot O. But the same applies to the other two endgroups, which says that three dots (people), namely A, F, G all have to be paired with O. This is impossible.

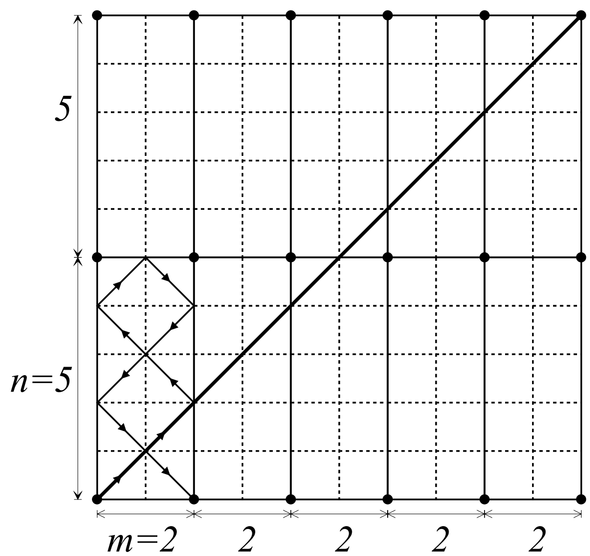
Note that it *is* true that everyone has three friends, with the even number 16 of partygoers, in this question.

8. **Millenniards** The numbers of bounces are:
3×5: 6 bounces; 4×5: 7 bounces; 5×5: 0 bounces; 6×5: 9 bounces; 7×5: 10 bounces; 8×5: 11 bounces; 9×5: 12 bounces; 10×5: 1 bounce.

It is pretty clear that $m \times n$ and $km \times kn$ tables give the same number of bounces, for $k = 1, 2, 3, \ldots$: the whole table is just expanded by a factor of k. For instance, 1×2 and 5×10 tables will have the same number of bounces (namely 1). So we can assume from now on that m, n have no common factor–for instance 10×35 will give the same number of bounces as 5×7, where we have taken out the common factor 5.

The trick now is to repeatedly 'open out' the table by reflection. In the figure, an $m \times n$ table is opened out a few times; the dashed line is the beginning of the actual path of the ball in the bottom left table, but the thick line is this path as it appears when you repeatedly reflect the table in one of its sides. The dashed line becomes a *straight* 45° line!





Suppose we reflect as many times as is needed to make an $mn \times mn$ square. The case of 2×5 is illustrated in the figure. The diagonal 45° line, which represents the path of the ball in the 'opened out' table, passes through the top right-hand corner of this square. When m and n have no common factor the line *will not pass through any corner of a $m \times n$ rectangle before this one.* (These corners are indicated by heavy dots in the example.) This is a crucial property, and any entrant for Senior Challenge who made an attempt to prove it was given extra credit. A proof is given at the end of this question.

The number of bounces is then the number of horizontal or vertical lines in the figure which are crossed by the diagonal line–not the dashed lines which represent the division of the rectangle into unit squares, but the solid lines which represent the edges of the table. Every crossing of an edge corresponds to a bounce. But the number of edges crossed is $m - 1$ horizontal ones and $n - 1$ vertical ones (1 horizontal line and 4 vertical lines in the example) before the ball ends in the top right-hand corner of the diagram, which means that it has entered a corner of the table. Thus the number of bounces is $m + n - 2$.

The rule is: Take out any common factor from m and n and work out $m + n - 2$ for this reduced pair of numbers. For example 25×35 would given 5×7 and then $5 + 7 - 2 = 10$ bounces.

From the diagram we can also read off the results (again taking out any common factor from m and n):
If n is even then the ball hits a corner of the table on the left side, i.e. top left corner.
If m is even then the ball ends on the right side, i.e. bottom right corner.
If m and n are both odd then the ball ends in the top right of the table.
(Note that m and n cannot both be even as they have no common factor!)

For 2000 bounces we need $m + n - 2 = 2000$, that is $m + n = 2002$. So we take any two numbers m and n which have no common factor and which add to 2002, and use a $m \times n$ table. We can also use a $km \times kn$ table for any such m, n and any k. I believe that there are 360 pairs (m, n) with $m + n = 2002$ and m, n having no common factor, counting (m, n) and (n, m) as the same pair. That's an awful lot of ways of making 2000 bounces!

Here is a proof that when m and n have no common factor, the diagonal line does not pass through any corner at coordinates (km, ln) before $k = n, l = m$, the corner at (mn, mn). Since the line is at 45° we have $km = ln$. But this implies that ln is a multiple of m. *Since m and n have no factor at all in common, this is only possible if l is a multiple of m.* So $l = m$ is the smallest possible value, and from this and $km = ln$ we get $k = n$.