Senior Challenge 1980

80-1. SUM OF THE YEAR

This is mostly a matter of intelligent trial and improvement. There are in fact 12 solutions, which come in four groups of three depending on the final column of the sums.

000	010	110	001	101	111	000	100	110	011	101	111
330	300	330	300	330	300	303	333	303	303	303	333
050	000	550	000	550	500	000	550	500	050	500	550
700	770	000	770	000	070	777	007	077	707	077	077
900	900	990	909	999	999	900	990	990	909	999	909

80-2. EXASPERATING

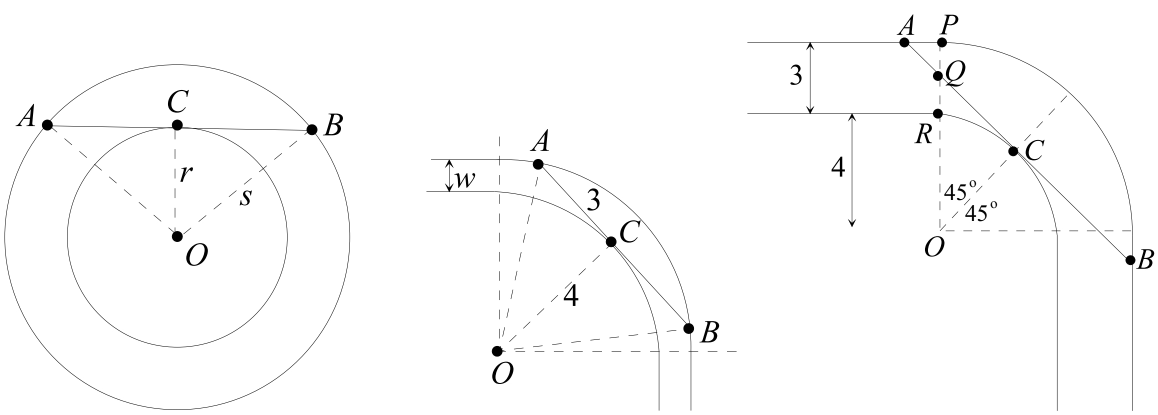
There is one word of each number of letters from 2 to 15. Note the following:

$$a^2 = bd,\ b^2c = ad \text{ imply } a^3 = abd = b \times b^2c = b^3c.$$

It follows that c must be a perfect cube, and so must be $2^3 = 8$, and $a = 2b$, from which it follows that $d = 4b$. So $(a, b, c, d) = (2b, b, 8, 4b)$. Assuming a, b, c, d are all different it follows quickly that $a = 6, b = 3, d = 12$ and the four words are 'length', 'and', 'fourteen', 'unconvincing'. But it is also possible to have $a = 4, b = 2, c = d = 8$ so that the four words are 'this', 'in', 'fourteen', 'fourteen'.

80-3. ROUND THE BEND

(a) From the left-hand figure, using Pythagoras's theorem, $CB^2 = s^2 - r^2$, so the longest straight line has length $AB = 2\sqrt{s^2 - r^2}$ metres.



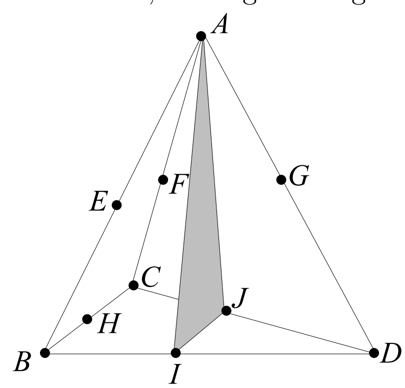
(b) In the middle figure, the critical position occurs when the pole lies wholly within the curved section, with the angle $AOB < 90°$. Using $AB = 6$ metres, $AC = 3$ metres Pythagoras's theorem gives $OA = 5$ metres, so the width w is 1 metre. For a pole of finite thickness to go round we need $w > 1$ metre.

(c) Now refer to the right-hand figure. The width is 3 metres and the flagpole is $11\frac{3}{4}$ metres long. Consider the symmetrical position as shown, which makes the distance AB as small as possible. If the pole does go round the bend then $AB > 11\frac{3}{4}$ metres. From the triangle QOC, using angle $QOC = 45°$, it follows that $QC = 4$ and hence $AB > 2QC$. Thus the angle $AOB > 90°$

and AB must encroach on the straight sections of the corridor as shown. Now $QO = 4\sqrt{2}$ metres, so that $PQ = PO - QO = 7 - 4\sqrt{2}$ metres. But triangle APQ is right-angled with $PQ = AP = 7 - 4\sqrt{2}$ metres. Using Pythagoras's theorem again, $AQ = PQ\sqrt{2} = 7\sqrt{2} - 8$ metres. Finally $AB = 2(AQ + QC) = 14\sqrt{2} - 8 = 11.799$ metres, which is *just* bigger than $11\frac{3}{4}$. A *very thin* pole will therefore go round the bend!

80-4. TRIANGULATION

There are in fact 114 triangles in all, 24 being equilateral. One can get these by simply listing all the types and working out how many there are of each. For example, for the *equilateral* triangles there are 4 like ABD, 4 like EGI, 4 like FHJ and 12 like AEG, making 24 altogether. On the other hand, any three points not in the same straight line make a triangle. The only straight line cases are the 6 such as AEB. Choosing three points from 10 we have 10 choices for the first point, 9 for the second and 8 for the third, so $10 \times 9 \times 8 = 720$ altogether, which would make $720/6 = 120$ triangles. (Three points A, B, C say make 6 triangles $ABC, BCA, CAB, BAC, ACB, CBA$: since the order of the points does not matter, we have counted the same triangle six times). But now we must eliminate the 6 'flat' triangles line AEB, leaving $120 - 6 = 114$ altogether.

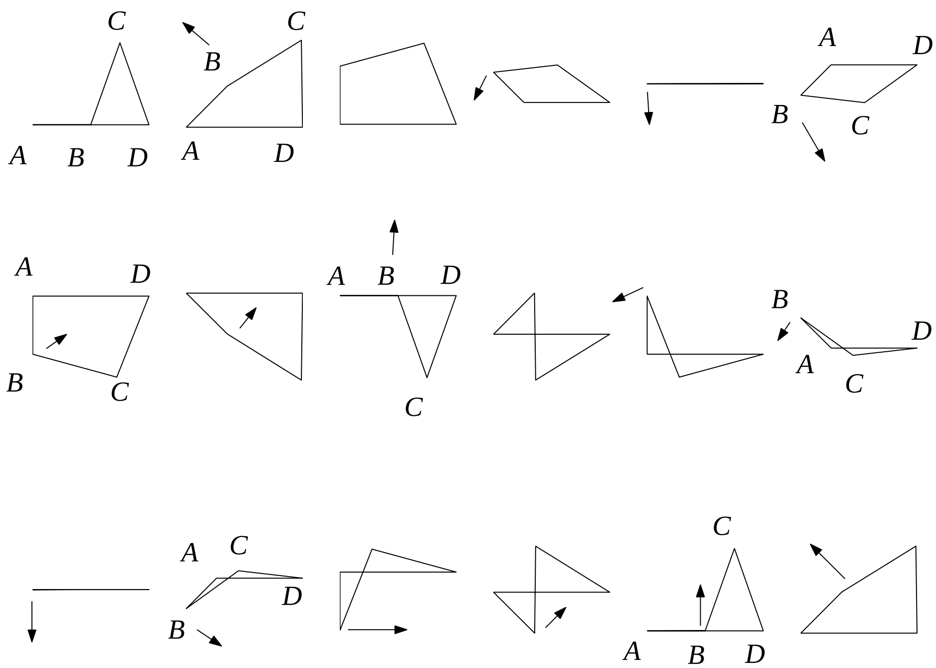


80-5. DEGREES OF FREEDOM

Naturally C moves on a circular arc, centred at D and radius 3; the question is how much of the circle is covered. In this particular case it is exactly *a semicircle*, for C will rotate clockwise until $ADC = 90°$ and ABC is straight ($3^2 + 4^2 = 5^2$ so the angle ADC *is* $90°$). After that, C starts to rotate anticlockwise. Later, B, A, C, D will be in a straight line with B to the left of A; this happens because $BA + AD = BC + CD = 6$. Then B and C continue to turn anticlockwise until C is below D and the angle ACD is again $90°$. After that, C starts to turn clockwise again until the original position is regained, with B between A and D.

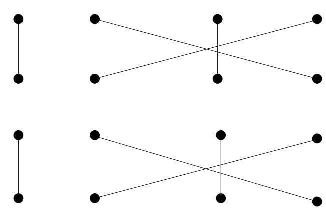
The figure on the next page shows several positions of the linkage, with the letters A, B, C, D on selected positions and the direction of movement of B indicated.

Note that this is a very special case, dependent on the exact measurements given. There is a thorough discussion of the movement of '4-bar linkages' in [1, p.228ff].



80-6. THEY ARE MAGIC!

The partner diagram for the lower magic square is shown on the right. There are 24 ways in which 0, 8, 10, 12 can be written in the main diagonal of either square (4 choices for the first digit, 3 for the second, 2 for the third and one for the last, making $4 \times 3 \times 2 \times 1 = 24$ ways). Some experimentation shows that, in the upper case, for every choice of arrangement for 0, 8, 10, 12, the positions of 15, 7, 5, 3 are then fixed, but there remain four possible arrangements of the remaining eight numbers, as, for example, in the set below, giving $24 \times 4 = 96$ such squares:



0	11	4	15		0	6	9	15
14	8	7	1		14	8	7	1
13	5	10	2		13	5	10	2
3	6	9	12		3	11	4	12

0	11	4	15		0	6	9	15
13	8	7	2		13	8	7	2
14	5	10	1		14	5	10	1
3	6	9	12		3	11	4	12

On the other hand, in only 8 of the 24 arrangements of 0, 4, 12, 14 is it possible in the upper case to complete the magic square at all. For each of these arrangements, however, the square can again be completed in four ways, giving $8 \times 4 = 32$ such squares.

Squares in the lower case are much fewer in number. In fact there is *none at all* with 0, 8, 10, 12 down the diagonal and only two with 0, 4, 12, 14 down the diagonal, namely

0	13	6	11
15	4	9	2
8	3	14	5
7	10	1	12

14	5	8	3
1	12	7	10
6	11	0	13
9	2	15	4

There is much information on magic squares in books on Mathematical Recreations. Two classics are [4] and [6]. In [4] a list is given on p.120 of 12 different types of square, two of which figure in the present problem. In [6] on pp.183–186 the 86 different sets of numbers between 1 and 16 adding up to 34 (equivalent to four different numbers between 0 and 15 adding up to 30) are listed and classified. See also the section on magic squares in [8]. Researchers should note that the binary notation is a natural one to use if 0 to 15 are taken in preference to 1 to 16, each number then being represented as a four-bit number, each bit being 0 or 1. It is then instructive to discover for which permutations of the digits of any four numbers adding up to 30 the sum remains 30.

There are 880 distinct magic 4×4 magic squares formed from the numbers 0 to 15, or 1 to 16, squares obtainable from each other by reflection or rotation being regarded as the same. The full list was first published by Frénicle in 1693.

Senior Challenge 1981

81-1. A CRACKER

Code digit	0	1	2	3	4	5	6	7	8	9
Equivalent	5	3	4	0	7	9	1	8	6	2

Underlining all code digits, we have $\underline{3} = 0$ (i.e. code 3 = plain 0) from $\underline{5} + \underline{3} = \underline{5}$. From $\underline{8} + \underline{7} = \underline{62}$ we get $\underline{6} = 1$ since the sum must be < 20. From $\underline{12} + \underline{8} = \underline{23}$ we get $\underline{2} = \underline{1} + 1$.

Also from $\underline{11} \times \underline{1} = \underline{55}$ we have $\underline{1} = 2$ or 3 since the right hand side has two digits. Suppose $\underline{1} = 2$. Then $\underline{2} = \underline{1} + 1 = 3$ and $\underline{5} = 4$. We then use $\underline{0} + \underline{9} = \underline{4}$ and $\underline{0} - \underline{9} = \underline{1} = 2$ to deduce (by adding) that $\underline{4}$ is even, and hence can only be 6 or 8 since other even digits are used up. But $\underline{4} = 6$ gives $\underline{0} = 4$: impossible as $\underline{5} = 4$, and $\underline{4} = 8$ gives $\underline{0} = 5, \underline{9} = 3$, again impossible as $\underline{2} = 3$.

So in fact $\underline{1} = 3$ and now we get $\underline{2} = 4, \underline{5} = 9$ and the other digits follow quickly.

81-2. A PARTY TRICK

This is best done using algebra. Let x and y be the original two numbers. Then the column is

$$\begin{array}{rcl} x & & \\ & & y \\ x & + & y \\ x & + & 2y \\ 2x & + & 3y \\ 3x & + & 5y \\ \longrightarrow 5x & + & 8y \\ 8x & + & 13y \\ 13x & + & 21y \\ 21x & + & 34y \end{array}$$

and adding these gives $55x + 88y$ which is 11 times the (arrowed) seventh line. (You may recognise the numbers 1, 1, 2, 3, 5, 8, 13, ... as being the *Fibonacci*[1] *numbers*.)

81-3. STOCK QUESTION

Suppose there are x boxes at £5 each, y boxes at £1 pound each and z bars at $10p = £\frac{1}{10}$ each. Then

$$\begin{aligned} x + y + z &= 100 \\ 50x + 10y + z &= 1000 \\ \text{so that } 40x &= 9z \end{aligned}$$

by taking 10 times the first equation from the second. Bearing in mind that x, y, z are positive whole numbers there are only two possibilities:

$$x = 9, \quad y = 51, \quad z = 40; \qquad \text{and} \quad x = 18, \quad y = 2, \quad z = 80.$$

81-4. LEFTOVERS

Since N is of the form $3a + 1$ it follows that $N + 2$ is a multiple of 3. Similarly from what we are told $N + 2$ is a multiple of 5 and 7, and this implies $N + 2$ is a multiple of $3 \times 5 \times 7 = 105$. It follows that the three lowest values of N are 103, 208 and 313, whose sum is $624 = 3 \times 208$.

[In fact if $N_1 + 2 = 105, N_2 + 2 = 2 \times 105, N_3 + 2 = 3 \times 105$ then adding gives $N_1 + N_2 + N_3 = 6 \times 105 - 6 = 3 \times (2 \times 105 - 2) = 3N_2$. Maybe you can see how to generalise this to the first r solutions for N, where r is odd.]

81-5. A LIKELY STORY!

The total cost of replacement after 5 years is I£1000 minus the second-hand value of the old car plus the sum of the running costs for 5 years:

$$1000 - 100 + (50 + 70 + 100 + 150 + 150) = 1420.$$

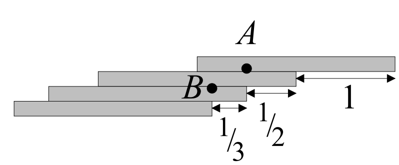
So the average cost per year in this case is $\frac{1420}{5} = \text{I}£284$. Similar calculations show the following:

Year	1	2	3	4	5	6	7	8	9	10
Total cost	450	670	920	1170	1420	1660	1930	2230	2530	2830
Average annual rate	450	335	$306\frac{2}{3}$	$292\frac{1}{2}$	284	$276\frac{2}{3}$	$275\frac{5}{7}$	$278\frac{3}{4}$	$281\frac{1}{9}$	283

From this it is clear that 7 years gives the smallest average annual rate.

81-6. CLIFFHANGER

The key here is to start from the *top*. The most the top domino can overhang the one beneath it is 1″. Furthermore the centre of gravity A of these two dominoes together is $\frac{1}{2}''$ from the right-hand end of the second domino. Take



these two and place them so that A is over the end of the third domino down, so that they just balance. The centre of gravity B of the top three dominoes will be $\frac{1}{3}''$ from the right-hand end of the third domino down. [Because $w \times \frac{2}{3} = 2w \times \frac{1}{3}$, w being the weight of the third domino and $2w$ that of the first and second dominoes.]

[1] Fibonacci (1170–1250) was also known as Leonardo of Pisa.

So far the overhang is $1 + \frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$ inches. Now placing the assembly of three dominoes so that its centre of gravity is over the end of the fourth domino down the overhang increases by $\frac{1}{4}''$ and this is the way that the process continues. The largest overhang altogether is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} = 2.829 \text{ inches.}$$

Notice that this means that the top domino completely overhangs the bottom one!

It can be shown that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n}$$

increases without bound as n increases—that is, the sum can be made as big as we please if we are willing to make n big enough[2]. This actually means that, in theory at least, the overhang could be made equal to a foot, or indeed 3 feet or any other amount you care to name. However, as you can check for yourself maybe[3], to get an overhang of 12 inches would require around 60,000 dominoes, and 3 feet = 36 inches would take more than 10^{15} dominoes. At a thickness of $\frac{1}{4}$ inch these dominoes would reach a height equal to about 67 times the distance from the earth to the sun![4]

Senior Challenge 1982

82-1. A QUESTION OF IDENTITY

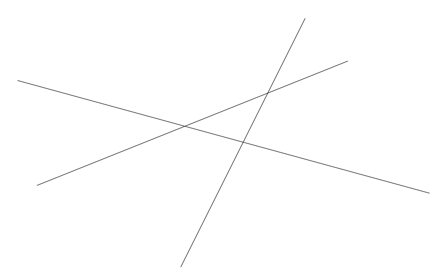
Suppose the required number has digits abc so that its numerical value is $100a + 10b + c$. Then according to the question,

$$\begin{aligned} a + b + c &= 12, \\ a - b &= b - c, \\ a &> b + c. \end{aligned}$$

Subtracting the second equation from the first gives $b = 4$ and therefore $a + c = 8$. Substituting into the third inequality gives $8 - c > 4 + c$, that is $c < 2$. But the number is odd, so c must be odd and is therefore 1. We now get $a = 7$ and the number is 741.

82-2. COLD CUTS

The first cut must obviously produce two pieces, and the second cut can divide each of these into two making four pieces so far. Likewise the third cut can divide each of the four pieces into two, making eight. An example is to divide by three cuts parallel to and halfway between the pairs of parallel faces of the cube, making eight cubes of half the side length of the original cube. So far we have eight pieces. Unfortunately the fourth cut cannot meet all eight pieces. The best we can do is for it to meet seven, making 15 pieces of ice cream in all. One way of seeing this is to think of the three lines in which the plane of the fourth cut is met by the planes of the other three cuts, as in the figure. Notice that the three lines divide the fourth



[2]We say that the 'harmonic series' $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is *divergent*. This is proved in books on infinite series or on 'analysis'.

[3]or maybe not.... It can be shown that for the sum to exceed M, we must have $n > e^{M-1}$. Here e is the 'base of natural logarithms', and is approximately 2.71828. As M gets larger, e^{M-1} gets truly enormous (exponential growth).

[4]The harmonic series is what is known as a *slowly diverging* series: it increases without bound provided you take enough terms but it is in no hurry.

plane into *seven* regions. These represent the seven pieces of ice cream which are split in two by the fourth cut, and hence the seven additional pieces created by that cut.

Of course we could continue in this way: with a fifth cut we consider the plane of that cut and the four lines in which it is met by the four planes of the first four cuts. A diagram similar to the figure will convince you that there are 11 regions in this plane. So at most 11 pieces can be cut in two by the fifth cut, making at most $15 + 11 = 26$ pieces are five cuts. Perhaps you can continue the series!

82-3. BOTTLE STOPPER

This is an example where *relative velocities* are useful. Imagine you are in another boat which is floating downstream alongside the bottle. After dropping the bottle, the woman travels away from you for a quarter of an hour, rapidly turns and heads back towards you. Now *from your point of view*, that is, measured relative to the water, her speed is the same for both journeys. so it will take her a quarter of an hour to return. She will therefore have been away for half an hour, during which time you and the bottle will have floated 2 miles downstream. Hence the water is flowing at the rate of 2 miles in half an hour or 4 m.p.h.

82-4. MISSING NUMBERS

Let us say that n digits were needed to number the houses on the odd-numbered side (ONS) of the road, and m digits for the houses already built on the even-numbered side (ENS). As each digit cost £0.50 and the total cost of all the digits was £42.50, we have $n + m = 85$. The difference between the ONS and ENS sides was £5.50, so $n - m = 11$. Solving these equations gives $n = 48$, which implies that the ONS must have 4 single-digit house numbers and 22 two-digit numbers, 26 houses in all. Hence the number of the last ONS house is 53.

Eventually there will be the same number of houses on each side of the road. Since the odd numbers start from 3 (not 1) the houses on both sides of the road will need the same number of digits to number them. When the block of houses missing from the ENS is built these houses will therefore, from the second equation above, need 11 digits to number them. As numbers 2 and 4 are already built the missing house numbers must accordingly be 8, 10, 12, 14, 16 and 18.

82-5. EGYPTIAN FRACTIONS[5]

Since 19 is a prime number, it must be a factor of at least one of the four denominators a, b, c, d in the required expression $\frac{17}{19} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. If we don't use $\frac{1}{2}$ then the largest sum we can get is $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{19} < \frac{17}{19}$, so we must use $\frac{1}{2}$, that is $a = 2$, say. So we are reduced to finding b, c, d whose reciprocals add to $\frac{17}{19} - \frac{1}{2} = \frac{15}{38}$.

If $\frac{1}{3}$ is present then we need to find c, d whose reciprocals add to $\frac{15}{38} - \frac{1}{3} = \frac{7}{114}$, with one or both denominators being a multiple of 19. Two solutions exist, giving

$$\frac{17}{19} = \frac{1}{2} + \frac{1}{3} + \frac{1}{18} + \frac{1}{171} = \frac{1}{2} + \frac{1}{3} + \frac{1}{19} + \frac{1}{114}.$$

Otherwise it can be shown by progressive elimination that there is only one other possible solution

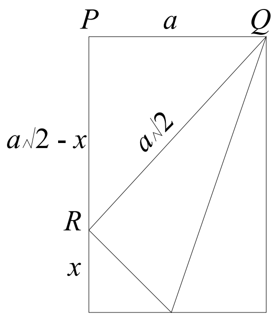
$$\frac{17}{19} = \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{532}.$$

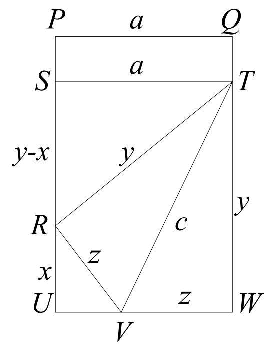
[5] An excellent source of reference for those interested in ancient—and modern—mathematics is the book [5]. Much of our information about ancient Egyptian mathematics (around 2000 BC) comes from the *Rhind Papyrus*, discovered in 1858 by an Englishman, Henry Rhind.

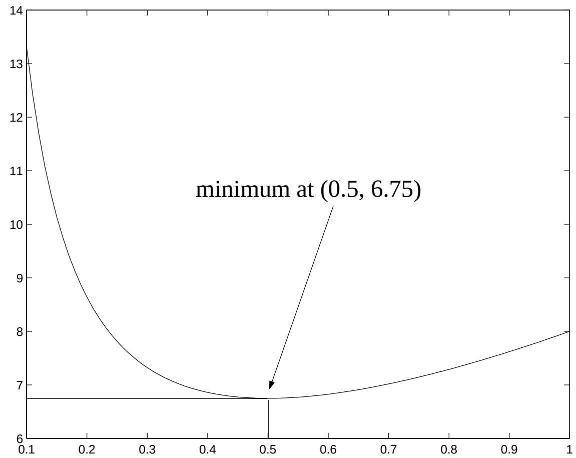
82-6. ORIGAMI ALGEBRA

If, in the figure in the question, $x > a$, the crease would no longer intersect the bottom edge of the paper. So $x \leq a$.

The left-hand figure shows the situation where the crease just reaches the top right-hand corner Q of the paper. In this case RQ is actually the whole of the right-hand edge folded over, so $RQ = a\sqrt{2}$. Applying Pythagoras's theorem to the right-angled triangle RPQ, it follows that $PR = PQ = a$. But $PR = a\sqrt{2} - x$, implying that $x = a(\sqrt{2} - 1)$, or, with $a = 21$ cm, $x = 8.70$ cm.







Here is a proof of the formula for c given in the question, referring to the middle figure, where ST is parallel to PQ. Let $TW = y$ and $VW = z$. Then $RT = y, RV = z, SR = y - x, UV = a - z$.

From triangle RST, using Pythagoras, $(y - x)^2 + a^2 = y^2$ and simplifying we get $y = (x^2 + a^2)/2x$. From triangle RUV, $x^2 + (a - z)^2 = z^2$, giving $z = (x^2 + a^2)/2a$. Finally from triangle TWV, $c^2 = y^2 + z^2$. Using the expressions just obtained for y and z we get

$$c^2 = \frac{(x^2 + a^2)^2}{4x^2} + \frac{(x^2 + a^2)^2}{4a^2} = \frac{(x^2 + a^2)^3}{4a^2x^2}.$$

To find the value of x^2 corresponding to the shortest crease, as a fraction of a^2, put $x^2 = ka^2$ in the formula for c^2:

$$c^2 = \frac{(ka^2 + a^2)^3}{4a^2ka^2} = \frac{a^2}{4}\frac{(k+1)^3}{k}.$$

Then, as $x \leq a$ we have $k \leq 1$, and we can plot the function $(k + 1)^3/k$ for values $0 \leq k \leq 1$ as in the right-hand figure. It is clear from the graph that this function attains its minimum value when $k = \frac{1}{2}$. (The minimum point can be checked using calculus if you know about this method.) When $k = \frac{1}{2}$ we get $x^2 = \frac{1}{2}a^2$, which on substituting in the formula for c shows that the shortest crease is of length $c = \frac{3}{4}a\sqrt{3}$ or 27.28 cm.

Finally, it is interesting to note that the shortest crease is obtained when $x = \frac{1}{2}a\sqrt{2}$, that is when the bottom right-hand corner of the paper is folded over to the point halfway up the left-hand edge.

Senior Challenge 1983

83-1. PROPER SIMPLETONS

If $\frac{a}{b}$ is a proper fraction in its simplest terms, then $b > a > 0$ and a, b share no common factors > 1. We are also given that, for each fraction, either a or b is a multiple of 4 but neither a nor b is a multiple of 3. There are only six proper fractions with single-digit numerator (top) and denominator (bottom) which satisfy all these conditions:

$$\frac{1}{8},\ \frac{5}{8},\ \frac{7}{8},\ \frac{4}{7},\ \frac{4}{5},\ \text{and}\ \frac{1}{4}.$$

The remaining condition is now imposed by adding 1 to each numerator and reducing to simplest terms where necessary:

$$\frac{1}{4},\ \frac{3}{4},\ 1,\ \frac{5}{7},\ 1,\ \text{and}\ \frac{1}{2}.$$

Clearly only two of the original fractions become equal when this is done: $\frac{7}{8}$ and $\frac{4}{5}$.

83-2. KEEP IT DARK!

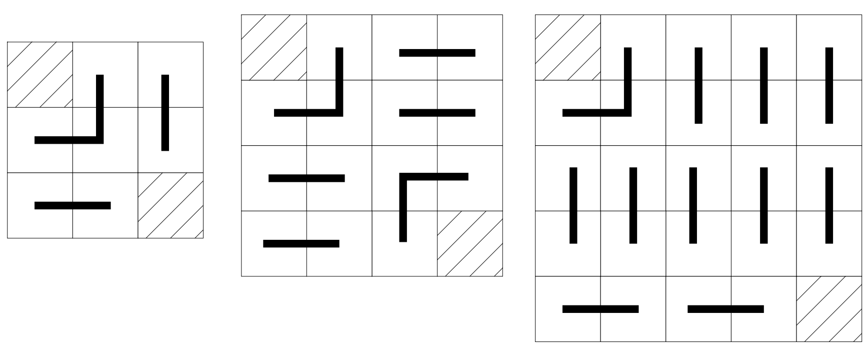
The Vino family keeps the small cupboard (S) for daily access to wine. When S becomes empty, they transfer bottles to it from the main stock held in the large cupboard (L). Each time S or L is opened, all bottles of wine inside are exposed to the light. Hence S or L cannot be opened to remove wine more than 11 times.

Let us assume that the family buys N bottles of wine. They drink 1 immediately, put 11 into S for consumption during the following 11 days, and store the remaining $N - 12$ bottles in L. When L is opened for more wine, its contents are exposed for the second time, so only 10 bottles can be transferred into S. Accordingly the family takes 11 from L, drinks 1 and puts 10 into S for use during the next 10 days, leaving $N - 12 - 11$ bottles in L. The next time L is opened only 9 bottles can be put into S and 1 drunk immediately, leaving $N - 12 - 11 - 10$ bottles in L, exposed 3 times.

The process is continued until L contains $N - 12 - 11 - \ldots - 3 - 2$ bottles, exposed 11 times. *Now*, when L is opened, only *one* bottle must remain—to be drunk right away as it has been exposed 11 times! Hence $N - 12 - 11 - \ldots - 3 - 2 = 1$, which gives $N = 78$. Therefore as the Vinos drink one bottle of wine each day, they need to buy wine every 78 days.

83-3. BLIND CORNERS

The figure shows the cases $n = 3, 4, 5$. Generally, if n is odd then put one triad round a blocked



corner and fill the other squares with dominoes, following the pattern in the figure. If n is even put a triad round each blocked corner and fill the rest with horizontal dominoes as in the figure for $n = 4$.

The are in general $n^2 - 2$ available squares. These cannot be covered using dominoes alone when n is odd because in that case $n^2 - 2$ is itself odd, so cannot be divided exactly by 2. When n is even, it helps to colour the board chessboard-fashion, with alternate black and white squares, the blocked squares being black say (note that they will always be the same colour). Hence the available squares include two more white squares than black ones, so dominoes, which always occupy a black and a white square, cannot cover the available squares.

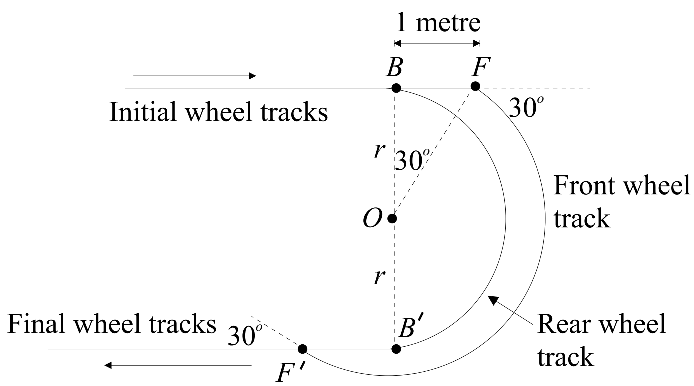
Finally, the available squares cannot be covered using triads alone because each triad occupies three squares and $n^2 - 2$ is never divisible exactly by 3. To see this last statement, note that n must have one of the three forms $3k, 3k+1$ or $3k-1$ for a whole number k. Then $n^2 - 2$ is respectively $9k^2 - 2 = 3(3k^2) - 2, 9k^2 + 6k - 1 = 3(3k^2 + 2k) - 1$, or $9k^2 - 6k - 1 = 3(3k^2 - 2k) - 1$. None of these three is a multiple of 3, proving the result.

83-4. COVER-UP

Assuming that you have first move, place the first counter in the centre of the board. Then reply to each of your opponent's moves by placing a counter at the same distance from the centre as their counter but diametrically opposite it. Symmetry dictates that there will always be a space left at the diametrically opposite position for you to play, and hence that you will inevitably have the last move of the game.

83-5. ON YER BIKE!

In the figure, B and F mark the centres of the back and front wheels respectively, when the front wheel turns suddenly through 30°. The front wheel then moves across the outer circle (radius OF) to F', and the back wheel moves around the inner circle (radius $OB = r$) to B'. Then the



front wheel turns back through 30° as shown. Since $BF = 1$ metre and angle BOF is 30°, we have $r = \sqrt{3}$, either by trigonometry or by thinking of OBF as half an equilateral triangle of side 2. Therefore the distance between the initial and final wheel tracks is $2r = 2\sqrt{3}$ metres.

83-6. SEEING STARS

After some experimentation, we come up with the following observations:

- Stars with $m = k$ and $m = n - k$ are of identical shape, for example if $n = 7$, then $m = 1$ and $m = 6$ (which is like $m = -1$, meaning go backwards one step) are the same shape, as are $m = 2, m = 5$ ('$m = -2$') or $m = 3, m = 4$ ('$m = -3$').

- If m and n share a common factor > 1 then the star produced splits into two or more smaller stars. In fact suppose $m = m_1k, n = n_1k$ where k contains all the factors common to m and n so that m_1 and n_1 have no common factor > 1. Then there are k stars, starting respectively with points $1, 2, \dots, k$ round the circle. For example starting with the first point round the circle, the star consists of points $1, 1+m, 1+2m, \dots, 1+n_1m$ since $n_1m = n_1km_1 = nm_1$ is a multiple of n and so we are back at the first point again having completed a star with n_1 points. All the stars have n_1 points and there are k of them since $n = n_1k$. For $m = 8, n = 12$ we have $m_1 = 2, n_1 = 3, k = 4$ and the points 1, 9, $17 = 5, 25 = 1$ form a star with $n_1 = 3$ points. The other stars with 3 points are 2, 10, $18 = 6$; 3, 11, $19 = 7$; and 4, 12, $20 = 8$.

- For $m = \frac{1}{2}n$ the figure isn't really a star but two identical straight lines.

In the table we have, for various values of n, the corresponding values of m which give *unsplittable* stars of different shapes.

n	m	Total number of stars
7	1 2 3	3
8	1 3	2
9	1 2 4	3
10	1 3	2
15	1 2 4 7	4

Finally, for $n = 42$, distinct unsplittable stars arise from values of $m < 21$ which have no factors in common with 42, that is $m = 1, 5, 11, 13, 17, 19$, giving a total of six stars.

Senior Challenge 1984

84-1. STRAIGHT TO THE POINT

Let set A be all possible scores with one dart. Then the set of possible scores with two darts—set B say—is obtained by adding any two numbers in A. Similarly the scores with three darts are obtained by adding any two numbers in A and B. The results are set out in the table below, with the two lowest impossible scores underlined in each case. Not that the 'bogey numbers'—the impossible scores—are not necessarily prime.

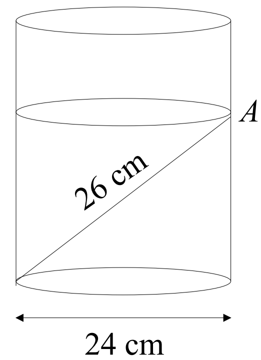
	Possible scores	Impossible scores
One dart only	1 to 20	
	21, 22, 24, 25, 26, 27, 28	23, 29
	30, 32, 33, 34, 36, 38, 39	31, 35, 37
	40, 42, 45, 48	41, 43, 44, 46, 47, 49
	50, 51, 54, 57, 60	52, 53, 55, 56, 58, 59
Two darts	2 to 100	103, 106, 109
	101, 102, 104, 105, 107, 108	112, 113, 115, 116
	110, 111, 114, 117, 120	118, 119
Three darts	3 to 160	163, 166, 169
	161, 162, 164, 165, 167, 168	172, 173, 175
	170, 171, 174, 177, 180	176, 178, 179

84-2. THE THREE SQUARES

Suppose Mrs Bear's room has x^2 tiles and Mr Bear's room y^2 tiles. Then $x^2 = N + 99$, $y^2 = N + 200$. Subtracting these equations, $(y - x)(y + x) = y^2 - x^2 = 101$. Since 101 is prime it must be that $y - x = 1, y + x = 101$. These give $y = 51, x = 50$ so that $N = x^2 - 99 = 2401$.

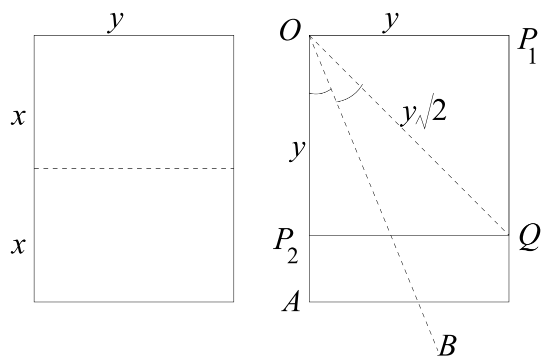
84-3. HIDDEN DEPTHS

This time the ursine household needs geometry instead of algebra. In the schematic diagram, we can assume that the spoon comes to rest with one end against the inner rim at the bottom of the pan and the other end touching the side at point A as shown. Using Pythagoras's theorem, the distance between A and the bottom of the pan is $\sqrt{26^2 - 24^2} = 10$ cm. This is the minimum depth of porridge needed to hide the spoon completely. As the radius of the pan is 12 cm, the minimum volume of porridge is $\pi \times (12)^2 \times 10 = 4523.9$ cu. cm.



84-4. ORIGAMI RATIO

Referring to the left-hand figure, we are given that $x/y = y/2x$, which simplifies to $2x^2 = y^2$, or $y = x\sqrt{2}$. Therefore the ratio of sides is $x\sqrt{2} : 2x$ or $1 : \sqrt{2}$. For the second part, refer to the right-hand figure. Fold the paper along OQ, that is, so that the top edge OP_1 lies along the left-hand edge. Let the corner P_1 reach the point P_2 on the left-hand edge, so $OP_1 = OP_2 = y$.



Applying Pythagoras's theorem, we get $OQ = y\sqrt{2}$, as shown. Now fold the paper again across the dotted bisector line OB of the angle AOQ, that is so that OQ falls along the left side of the rectangle. If the ratio of sides is indeed $\sqrt{2}$ then Q will fall at A.

84-5. ORBITERS

The planet P_n takes n years to complete its orbit, so after t years from the starting position when all planets are lined up, P_n will have made t/n orbits. Consider a pair of planets P_n and P_m. If $n < m$, then P_n will be moving around faster than P_m and will overtake P_m at time t_{mn} say. At this moment P_m will have completed say x orbits, where $t_{mn} = mx$. Since P_n is just overtaking P_m, it must have made $x + 1$ orbits, so $t_{mn} = n(x + 1)$. Substituting for x from the first equation in the second gives

$$t_{mn} = \frac{nm}{m - n} \text{ years}$$

as the first time when P_n and P_m line up on the same side of the star.

We can tabulate these results as follows, taking all the values for m and n.

Planets	P_1P_5	P_1P_4	P_1P_3	P_1P_2	P_2P_5	P_2P_4	P_2P_3	P_3P_5	P_3P_4	P_4P_5
t_{mn}	$1\frac{1}{4}$	$1\frac{1}{3}$	$1\frac{1}{2}$	2	$3\frac{1}{3}$	4	6	$7\frac{1}{2}$	12	20

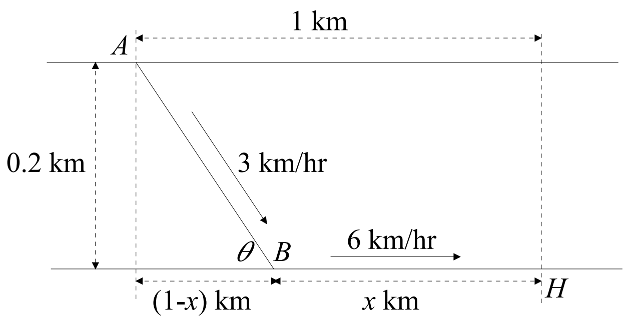
The pairs of planets will line up again on the same side of the star at times which are multiples of those given in the table. Therefore if P_n and P_m first line up at time t_{mn} and say P_q and P_r first line up at time t_{qr} then the *pair* P_n, P_m will first line up with the *pair* P_q, P_r at a time which is the smallest number which is a multiple of both t_{mn} and t_{qr}. For example, P_1, P_5 will line up with P_2, P_4 after the smallest number of years which is a multiple of both $1\frac{1}{4} = \frac{5}{4}$ and 4. Clearly this number is 20.

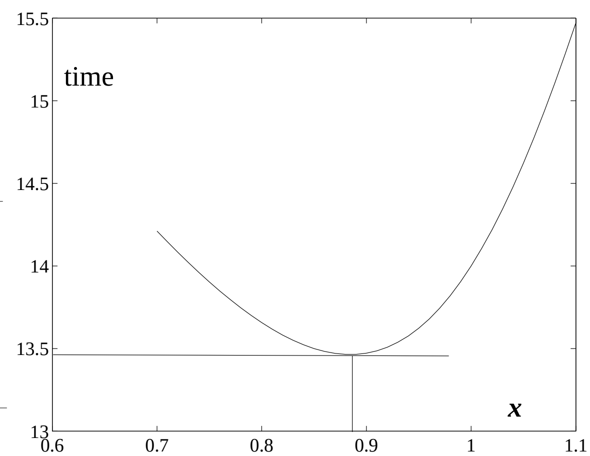
To find the first time when *three* planets line up we look at the table for pairs which include those planets. On doing this, it is not difficult to see that the first possible three-fold line-up occurs between P_1, P_2 and P_4 when the pairs $(P_1, P_2), (P_1, P_4)$ and (P_2, P_4) line up for the first time. The smallest number which is a multiple of 2, $1\frac{1}{3} = \frac{4}{3}$ and 4 is 4, so this line-up happens after 4 years.

Using similar arguments, we can find the first line-up of *four* planets involves P_1, P_2, P_3 and P_4, and this occurs after 12 years, and the first line-up of all five planets occurs after 60 years, the smallest number which is a multiple of all 11 numbers in the table.

84-6. HOME JAMES!

The left-hand figure shows the route taken by James Bond: he swims at 3 km/hr from A to B and then lopes wearily at 6 km/hr from B to his house at H. If B is directly opposite A across





the river, $AB = 0.2$ km and $BH = 1$ km. Therefore the total time taken is

$$\frac{60 \times 0.2}{3} + \frac{60 \times 1}{6} = 4 + 10 = 14 \text{ min.}$$

In general, when $BH = x$ km, then using Pythagoras's theorem we find

$$AB = \frac{1}{5}\sqrt{1 + 25(1-x)^2}$$

and the total time taken is

$$\frac{60 \times AB}{3} + \frac{60x}{6} = 4\sqrt{1 + 25(1-x)^2} + 10x \text{ min.}$$

The right-hand figure shows the graph of this function, and the minimum is a little below 13.5 min., at about $x = 0.88$. (Using calculus it can be shown that the exact answer is $10 + 2\sqrt{3}$, which is approximately 13.464 min, when $x = 1 - (1/5\sqrt{3})$, which is approximately 0.8845.)

The tangent of the angle θ is $0.2/(1-x)$ which comes to about 1.67, and this gives $\theta = 59.03°$. (The exact answer is $\tan\theta = \sqrt{3}$, which implies $\theta = 60°$ exactly.)

Senior Challenge 1985

85-1. FLIP START

Let x and y be any two numbers chosen from 1 to 9. Then answer A is $x + y$. The required two-digit numbers are $n = 10x+y$ and $m = 10y+x$. Hence the answer B is $n+m = 11x+11y = 11(x+y)$, and the quotient B/A always equals 11. [Extra: suppose $x > y$. Then it can be shown that the sum of the digits of the number $n - m$ always equals 9.]

85-2. BICYCLE MADE FOR ONE

Suppose Bill leaves the bicycle beside the road after riding it for x km at 12 km/hr. He will then have to walk home, a distance of $16 - x$ km, at 5 km/hr. The total time taken is

$$T_B = \frac{x}{12} + \frac{16-x}{5} \text{ hours.}$$

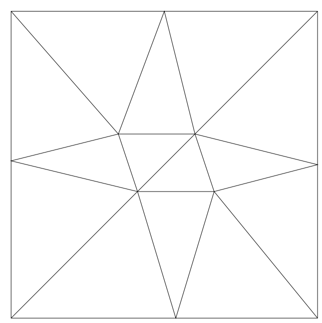
Meanwhile, Alan, walking at 4 km/hr, covers the distance x to the bicycle in $x/4$ hours and then rides the $16 - x$ km home at 10 km/hr. The total time taken is

$$T_A = \frac{x}{4} + \frac{16-x}{10} \text{ hours.}$$

As they arrive home at the same time, $T_A = T_B$, and solving this equation gives $x = 6$ km. Hence they both take $2\frac{1}{2}$ hours to get home, during which Bill rides the bicycle for $x/12 = \frac{1}{2}$ hr.

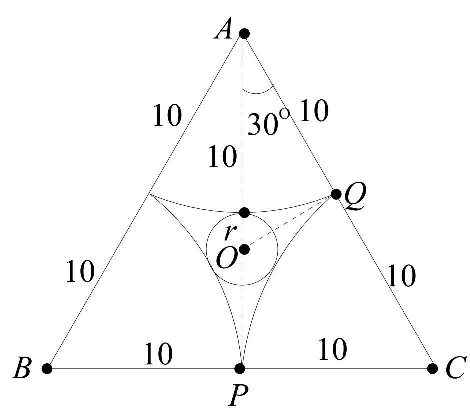
85-3. A CUTE CAKE

There are many possible ways in which Sheila can cut the cake into 14 acute-angled triangular pieces. The figure shows one possibility. [If you allow pieces to include complete sides of the cake then in fact it is possible to reduce the number of acute-angled triangles. We think that 10 is the minimum number—can you see how to achieve this?]



85-4. GRAZE ELEGY

In the figure, AOQ is half of an equilateral triangle, because angle AOQ is $60°$ and angle OQA is a right-angle. So $OQ = \frac{1}{2}AO = \frac{1}{2}(10 + r)$. Pythagoras's theorem applied to triangle AOQ gives $(10+r)^2 = (\frac{1}{2}(10+r))^2 + 10^2$, and solving this gives $\sqrt{3}(10+r) = 20$, so $r = (20/\sqrt{3})-10 = 1.547$ m approximately. So the diameter of the largest flowerbed Ms Gardner can safely have in the centre of the lawn is $2r = 3.094$ m approximately. [This can also be done by trigonometry: $AQ = AO\cos 30°$ so $10 = (10 + r)\sqrt{3}/2$.]



Similarly, $AP^2 = AC^2 - PC^2 = 300$ so $AP = 10\sqrt{3}$ m. The area of triangle ABC is $\frac{1}{2}BC \times AP = \frac{1}{2}20 \times 10\sqrt{3} = 173.205$ sq m approximately. The total area grazed by sheep is $3 \times \frac{1}{6}$ area of circle radius 10 m, since the circular sector centre C with arc from P to Q, for example, has an angle at C of $60°$ which is one-sixth of the full angle $360°$. The total area is then $3 \times \frac{1}{6} \times \pi(10)^2 = 157.080$

sq m approximately. The area of the circular flowerbed is $\pi r^2 = 7.518$ sq m approximately. The area left to mow is therefore

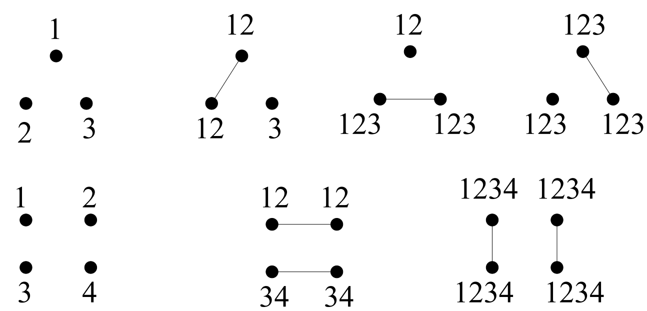
Area of lawn – Area grazed by sheep – Area of circular flowerbed = 8.607 sq m approximately.

85-5. SMART ALEC

The solution to this problem has nothing to do with probability! The colour of the last smartie is determined precisely by the original contents of the bag. The crux of the matter is that the number of *red* smarties in the bag can only decrease in *twos*. So if the bag originally contains an *odd* number of red smarties they will decrease until only one red one is left, which if removed with a green one will have to be returned to the bag. Hence in this case the last smartie must be *red*. However, if there is an *even* number of red ones in the bag to begin with, they will *all* eventually be removed, implying that the last smartie in this case has to be *green*.

85-6. I WANT RESULTS!

If three football matches had taken place, then the upper row of diagrams show that only three telephone calls are needed to share the results. Individual grounds are represented by dots and phone calls by lines. The results known at each ground are indicated by numbers alongside the



dots. For four matches, the lower row of diagrams shows the sequence involving the minimum number of calls. Clearly only four calls are needed.

For five matches, manager number 5 tells one of the other four, say number 1, his result. This is then shared amongst the group of four with their own results, four calls being made following the above sequence. Then number 1 communicates all the results back to number 5. Hence six calls are sufficient in this case.

For six matches, manager number 6 tells one of the other five his result. They then make six calls to share all the results amongst themselves as above. Finally all the information is telephoned back to number 6, making eight calls in all.

It is now becoming clear that for each extra ground *two* more calls are needed. So for seven matches, ten calls will do. In general, the number for $n \geq 4$ matches is $2n - 4$, since this fits for $n = 4$ and goes up by 2 when n goes up by 1. Note that $n = 3$ is the exception here: it needs 3 calls, not the number given by the formula.

Senior Challenge 1986

86-1. TIME TO START

During any 12-hour period the 'minutes' hand (M) travels around the clock face 12 times, whereas the 'hours' hand (H) goes round only once. Therefore in 12 hours M must overtake H eleven times. Between each of these coincidences M and H are at right-angles twice. Hence in 24 hours M and H are at right-angles at $2 \times 11 \times 2 = 44$ equally spaced times.

It follows that this event occurs every 24/44 hours, that is every $32\frac{8}{11}$ minutes. But we know that M and H are at right-angles at 3pm. Therefore they must be at right-angles $32\frac{8}{11}$ minutes earlier, that is at $2.27\frac{3}{11}$ pm. To the nearest second this is 2.27 and 16 seconds.

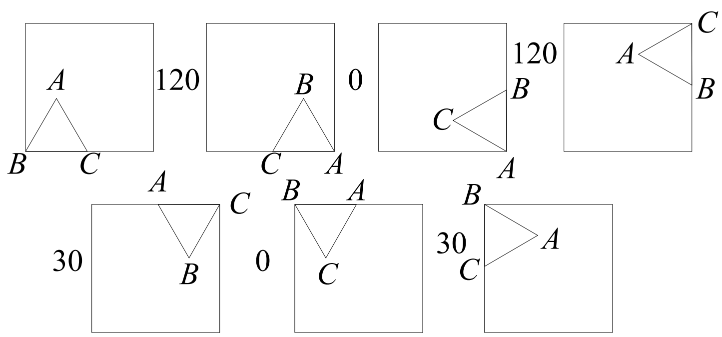
86-2. CALENDAYS

Calling January 1st 'day X' of the week, the first days of the remaining months of the year are
non-leap year: Feb $X+3$; Mar $X+3$; Apr $X+6$; May $X+8=X+1$; Jun $X+4$; Jul $X+6$; Aug $X+9=X+2$; Sep $X+5$; Oct $X+7=X$; Nov $X+3$; Dec $X+5$
leap year: Feb $X+3$; Mar $X+4$; Apr $X+7=X$; May $X+2$; Jun $X+5$; Jul $X+7=X$; Aug $X+3$; Sep $X+6$; Oct $X+8=X+1$; Nov $X+4$; Dec $X+6$.
Thus in a non-leap year the months of February, March and November begin on day $D = X+3$ and January 1st the following year begins on day $X+1$. In a leap year the months of January, April and July begin on day $D = X$ and January 1st the following year begins on day $X+2$.
This enables us to form the table, where L denotes leap year.

Year	X	D	
1986	Wed	Sat	← Given D = Sat for 1986
1987	Thurs	Sun	
1988 L	Fri	Fri	
1989	Sun	Wed	
1990	Mon	Thurs	
1991	Tues	Fri	
1992 L	Wed	Wed	
1993	Fri	Mon	
1994	Sat	Tues	
1995	Sun	Wed	
1996 L	Mon	Mon	
1997	Wed	Sat	← X = Wed and D = Sat for 1997
1998	Thurs	Sun	
1999	Fri	Mon	
2000 L	Sat	Sat	
2001	Mon	Thurs	
2002	Tues	Fri	
2003	Wed	Sat	
2004 L	Thurs	Thurs	
2005	Sat	Tues	
2006	Sun	Wed	
2007	Mon	Thurs	
2008 L	Tues	Tues	
2009	Thurs	Sun	
2010	Fri	Mon	← D is Monday only in 2010 between 2000 and 2019
2011	Sat	Tues	
2012 L	Sun	Sun	
2013	Tues	Fri	
2014	Wed	Sat	
2015	Thurs	Sun	
2016 L	Fri	Fri	
2017	Sun	Wed	
2018	Mon	Thurs	
2019	Tues	Fri	

86-3. TURNING POINTS

The figure shows the first seven positions of ABC, the numbers between frames being the numbers of degrees which the point A turns through, always on a circle of radius 2 cm. By the final position A has turned through 300° and has therefore travelled a total distance of $\frac{300}{360} \times 2\pi \times 2$ cm, which is $\frac{10}{3}\pi$ cm. *Now the seventh frame is the same as the first frame except*



that the square and the triangle have both been rotated clockwise through $90°$. Hence to get back to the original state the above procedure has to be repeated three more times: 24 frames after the first one restores the original state. This means that the point A has to move in all a distance of $4 \times \frac{10}{3} \times \pi$, which is 41.89 cm to two decimal places.

Each complete circuit which results in the triangle being in the bottom left of the square and attached to the bottom edge requires eight moves. Note that as A is in a different location on the triangle at the end of circuits one and two, the total angle A turns through is actually different for each circuit, in fact $420°, 450°$ and $330°$ for circuits one, two and three. These add to $1200°$, agreeing with 4×300 as in the first calculation.

86-4. SMART ARTIST

This problem can be tackled by calculating the area of canvas painted by the mathematical artist with each colour, and then adding the results until over half the area is covered. However, a more efficient approach is to consider how much of the canvas is *left unpainted* at each stage. Suppose the original area is A. First of all the artist paints one square red, leaving $\frac{8}{9}A$ unpainted. Then with yellow he paints eight squares, leaving $\frac{8}{9} \times \frac{8}{9}A$ unpainted. Then 8×8 squares are painted blue, leaving $(\frac{8}{9})^3 A$ unpainted. Similarly, after the fifth colour the remaining area is $(\frac{8}{9})^5 A = 0.555A$ and after the sixth colour $(\frac{8}{9})^6 A = 0.493A$ remains. Hence to cover over half the canvas he needs six colours.

When he has dealt with *all* the sixth set of squares. the total number painted is $1 + 8 + 8^2 + 8^3 + 8^4 + 8^5 = 37449$. (However, it can be shown that, if the artist uses just enough of the sixth colour to complete half the area, then only 33873 squares need to be painted altogether.)

Extra: when n colours have been used, the artist has painted a total area

$$A_n = \left(1 - \left(\frac{8}{9}\right)^n\right) A.$$

Now, the larger n becomes, the closer $(\frac{8}{9})^n$ approaches to zero. Hence, assuming he has an unlimited range of colours—and paintbrushes small enough to paint them!—he can make A_n as close to A as he likes. In mathematical language we say

$$\text{Limit}_{n\to\infty} A_n = A.$$

86-5. WEIGHT FOR IT![6]

This is actually a cunningly disguised exercise in 'base 3 arithmetic', that is, expressing any number as a sum of powers of 3, for example (and remembering $3^0 = 1$)

$$34 = 1 + 2 \times 3 + 3^3, \quad 20 = 2 + 2 \times 3^2, \quad 25 = 1 + 2 \times 3 + 2 \times 3^2.$$

The reason for the base 3 is that for each weight there are *three* choices: put it in the pan not containing the parcel, put it in the pan containing the parcel or don't use that weight at all. In the expression of a number in base 3, the 'coefficients', or 'digits' in front of the powers of 3 are always 1, 2 or 0: for 34 these 'digits' are 1, 2, 0, 1; for 20 they are 2, 0, 2, 0 and for 25 they are 1, 2, 2, 0.

We can use the weights $1, 3, 3^2 = 9, 3^3 = 27$ to weigh any amount up to $1 + 3 + 3^2 + 3^3 = 40$, as follows. What we do is to write numbers in base 3 but whenever there is a '2' in front of a power of 3 we replace 2 by $3 - 1$, as in these examples:

$$34 = 1 + 2 \times 3 + 3^3 = 1 + (3 - 1) \times 3 + 3^3 = 1 - 3 + 3^2 + 3^3,$$

so 34 pounds can be weighed with 1, 9 and 27 pound weights in the pan opposite to the parcel and 3 in the same pan as the parcel.

$$20 = 2 + 2 \times 3^2 = (3 - 1) + (3 - 1) \times 3^2 = -1 + 3 - 3^2 + 3^3,$$

so 20 pounds can be weighed with 3, 27 in the opposite pan and 1, 9 in the same pan as the parcel.

$$25 = 1 + 2 \times 3 + 2 \times 3^2 = 1 + (3 - 1) \times 3 + (3 - 1) \times 3^2 = 1 - 3 + 3^2 - 3^2 + 3^3 = 1 - 3 + 3^3,$$

so 25 pounds can be weighed with 1 and 27 in the pan opposite to the parcel and 3 in the same pan as the parcel.

In this way every weight up to $1 + 3 + 3^2 + 3^3 = 40$ can be handled, but notice that we can't go beyond 40, since for example

$$\begin{aligned} 41 &= 2 + 3 + 3^2 + 3^3 = (3 - 1) \times 1 + 3 + 3^2 + 3^3 = -1 + (3 - 1) \times 3 + 3^2 + 3^3 \\ &= -1 - 3 + (3 - 1) \times 3^2 + 3^3 = -1 - 3 - 3^2 + (3 - 1) \times 3^3 = -1 - 3 - 3^2 - 3^3 + 3^4, \end{aligned}$$

and we need a weight of $3^4 = 81$ to weigh 41 pounds by this scheme. The penalty of changing a '2' to an '–1' is that the next higher power of 3 is increased by 1, and in a number greater than 40 this leads to a power of 3 greater than 3^3.

Here is the table for n = parcel weight as far as 20, showing on the left the expression for n with powers of 3 and 'digits' 0, 1 and 2, and, on the right, the expression with powers of 3 and 'digits' 0, 1 and –1. Thus + means put the weight in the pan without the parcel and – means put the weight in the pan with the parcel. We hope you agree the pattern of + and – signs is quite easy to continue!

[6] Unusually for Senior Challenge questions, this is actually quite a famous mathematical problem, and you can read accounts of it in a number of places, for example in the classic book [8], under 'Bachet's weight problem'.

n	1	3	9	27	1	3	9	27
1	1				+			
2	2				−	+		
3		1				+		
4	1	1			+	+		
5	2	1			−	−	+	
6		2				−	+	
7	1	2			+	−	+	
8	2	2			−		+	
9			1				+	
10	1		1		+		+	
11	2		1		−	+	+	
12		1	1			+	+	
13	1	1	1		+	+	+	
14	2	1	1		−	−	−	+
15		2	1			−	−	+
16	1	2	1		+	−	−	+
17	2	2	1		−		−	+
18			2				−	+
19	1		2		+		−	+
20	2	2			−	+	−	+

86-6. PRESTIDIGITATION

When a number n uses all the digits 1 to 9 the sum of its digits is $1+2+3+\ldots+9=45$. You can actually tell straight away that because the *sum* of the digits of n is divisible by 9, so also is n itself divisible by 9. This is a beautiful and useful rule; the proof is at the end of this solution.

There is a similar and useful rule that if the sum of the digits of a number is divisible by 3, then the number is divisible by 3. This follows in the same way from the last expression for n in the above argument.

Of course the problem can be solved by trial and error, but this is unlikely to show conclusively that there is *only one* possible 9-digit number satisfying all the conditions. The argument below, though long, does actually show this fact.

We shall use the standard notation $p|q$ to mean 'p divides exactly into q', which means the same thing as 'q is a multiple of p'.

Write the number n which we seek in the usual way as $abcdefghi$, that is $n = i + h \times 10 + g \times 10^2 + \ldots + a \times 10^8$. Similarly we write ab for the number formed by the two leftmost digits, etc.

(1) $2|ab$ implies that b is even.
(2) $3|abc$ implies (as above) that $a+b+c$ is a multiple of 3.
(3) $4|abcd$ implies that $4|cd$ and d is even.
Notice also that $5|abcde$ implies that $e=5$ since there are no digits 0 in n.
(4) $6|abcdef$ implies that f is even
(5) $6|abcdef$ implies that $a+b+c+d+e+f$ is a multiple of 3, so by (2) so is $d+e+f$.
(6) $8|abcdefgh$ implies that $8|fgh$ and h is even.

Now since b,d,f,h are all even and $e=5$ it follows that
(7) a,c,g,i must all come from the set $\{1,3,7,9\}$.

Using (3) and (7) we deduce that
(8) d is 2 or 6.

Using (4) and (6) we deduce that $8|gh$, and now (7) implies
(9) gh is one of 16, 32, 72, 96.

Case 1: $gh = 16$. Then (8) gives $d = 2$ and (5), (4) give $f = 8$, so (1) gives $b = 4$. Hence $a+b+c = a+c+4$ where a, c come from $\{3, 7, 9\}$ using (7) as $g = 1$. Therefore $a+b+c = 14$ or 16 or 20 which are all impossible from (2). This rules out $gh = 16$.

Case 2: $gh = 32$. Then from (8) we get $d = 6$, so $d + e + f = 11 + f$ so from (5) we have $f = 4$ and from (1) $b = 8$. Hence $a+b+c = a+c+8$ where a, c come from $\{1, 7, 9\}$ using (7). Therefore from (2) we get a, c come from $\{1, 9\}$ or $\{7, 9\}$ and the possibilities for $abcdefg$ are 1896543, 9816543, 7896543 and 9876543. However none of these is divisible by 7, so this rules out $gh = 32$.

Case 3: Suppose $gh = 96$. Then (8) gives $d = 2$, (5) gives $f = 8$ and (1) gives $b = 4$. But (7) implies that a, c come from $\{1, 3, 7\}$ and $a+b+c = a+c+4$ so (2) shows that in fact a, c come from $\{1, 7\}$, and now the possibilities for $abcdefg$ are 1472589 and 7412589. Neither of these is a multiple of 7 so this rules out $gh = 96$.

Case 4: Suppose $gh = 72$. Then (8) gives $d = 6$, (5) gives $f = 4$ and (1) gives $b = 8$. Next, $a+b+c = a+c+8$ where a, c come from $\{1, 3, 9\}$ using (7). So (2) implies a, c come from $\{1, 3\}$ or $\{1, 9\}$ so the possibilities for $abcdefg$ are 1836547, 3816547, 1896547 and 9816547. Of these, only the second is a multiple of 7. Since $h = 2$ we know finally that $i = 9$ and the only number to satisfy all conditions is 381654729.

Proof of the rule for multiples of 3 and 9

Suppose n has digits $a_0, a_1, \ldots a_k$, so that

$$n = a_0 + a_1 \times 10 + a_2 \times 10^2 + \ldots + a_k \times 10^k,$$

where of course each digit is one of the numbers 0, 1, 2, up to 9, and written in the normal way n would appear as '$a_k \ldots a_2 a_1 a_0$'. For example $249 = 9 + 4 \times 10 + 2 \times 10^2$. Then

$$n = a_0 + a_1 \times (9+1) + a_2 \times (9+1)^2 + \ldots a_k \times (9+1)^k.$$

Now $(9+1)^2 = 11 \times 9 + 1$, $(9+1)^3 = (9+1) \times (11 \times 9 + 1) = (99 + 11 + 1) \times 9 + 1 = 111 \times 9 + 1$ and so on. That is, the above expression for n can be written

$$n = a_0 + a_1 + a_2 + \ldots + a_k + \text{ some multiple of 9.}$$

In particular, if $a_0 + a_1 + \ldots a_k$ is a multiple of 9, then n is a sum of two multiples of 9, and hence is itself a multiple of 9.

Senior Challenge 1987

87-1. INTO GEAR!

We'll use notation R: roomy; M: middling; S: small.

Coat sizes Denzil and Dayglo > Dorrit implies that:
(i) Dorrit S with Denzil R or M and Dayglo R or M *or*
(ii) Dorrit M with Denzil R and Dayglo R

Hat sizes Dayglo > Denzil and Dorrit implies that:
(iii) Dayglo M with Denzil S and Dorrit S *or*
(iv) Dayglo R with Denzil M or S and Dorrit M or S.

Dayglo's hat size = Dorrit's coatsize implies both must be M from the above. Hence (ii) and (iii) apply: Denzil buys a roomy coat and a small hat.

87-2. FROM BAD TO WURST

We'll use notation G: good Bad; B: bad Bad; b: best Wurst; w: worst Wurst.

In a G, strings of sausages look like ...wbbbwbbbwbbbw... .
In a B, they look like ...wbwbwbwbwb... .

My *first* purchase was a string containing three b. This would look like:
G: bbb, bbbw, bbwb, bwbb, wbbb or wbbbw. That is, 3, 4 or 5 sausages in a G-string.
B: bwbwb, bwbwbw, wbwbwb or wbwbwbw. That is, 5, 6 or 7 sausages in a B-string.

Then I returned to the *same* shop to buy a string consisting of the *same* number of sausages as before. If G, the possibilities are:

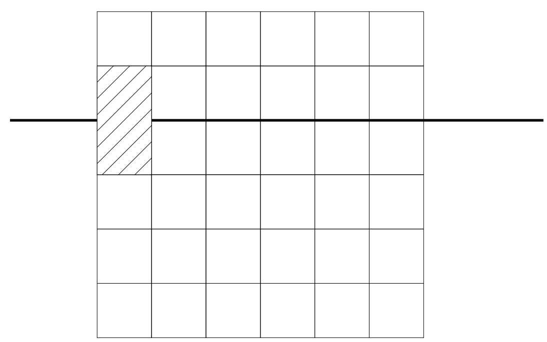
3 in a string: bbb, bbw, bwb or wbb. That is, two or three b.
4 in a string: bbbw, bbwb, bwbb or wbbb. That is, three b.
5 in a string: bbbwb, bbwbb, bwbbb or wbbbw. That is, three or four b.

On the other hand the B possibilities are:

5 in a string: bwbwb or wbwbw. That is, two or three b.
6 in a string: bwbwbw or wbwbwb. That is, three b.
7 in a string: bwbwbwb or wbwbwbw. That is, three or four b.

Hence whether I visited a good Bad or a bad Bad for both purchases my second string could have contained two, three or four best Wurst.

87-3. SPLITTING HEADACHE



Dominoes are to be laid to cover the 36 squares of the 6×6 grid. There are five horizontal and five vertical internal lines along which the resulting pattern may separate into two rectangles. Suppose that, as in the figure, there was only one domino 'blocking' one of these lines, which is extended out beyond the grid in the figure. This means that the *odd* number of squares above this line must be entirely filled by dominoes which do not cross the line. This is plainly impossible as each domino covers two squares. The result of this is that each line must be blocked by at least two dominoes, and with ten lines to block this means 20 dominoes are needed, whereas we only have 18. So it is impossible to block all the lines.

It is interesting to carry this on to larger $n \times n$ grids with n even. For example, with an 8×8 grid, there are 14 internal lines and each takes at least two dominoes to block it, requiring 28 dominoes. But this time we have 32 dominoes to play with so maybe all lines can be blocked! In fact this is possible, and you may enjoy finding a solution.

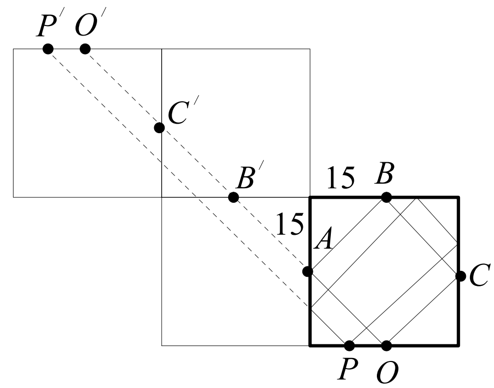
87-4. RING THE CHANGES!

Let the telephone number of house number r be $t + r$. Then $t + r$ is divisible by r. This means that t itself must be divisible by all the values of r: 1, 2, 3 up to 16. Thus t is divisible by the least common multiple of all these numbers, which is $2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 = 720720$.. That is, $t = 720720N$ where N is a whole number.

Now, for $t, t+1, t+2, \ldots, t+16$ to be 7-digit numbers, N has to lie in the range $2 \leq N \leq 13$.

The telephone number of house number 13 is $720720N + 13$. This has to be divisible by the new house number which is 17. By trial, only one number in the above range satisfies this criterion, $N = 11$. Hence the telephone number of the new house number 17 is $720720 \times 11 + 13 = 7927933$.

87-5. SQUARE ROUTE

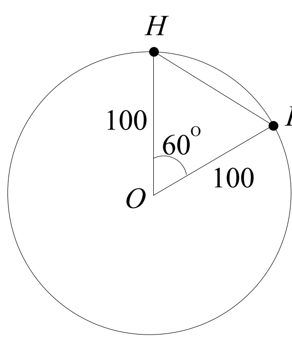


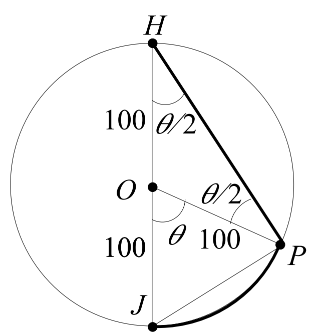
This is an example where the clearest way of seeing the shortest route is to reflect the playground repeatedly in a side, as in the figure, where O is Cynthia's starting point at the midpoint of a side and A, B, C are midpoints of the other sides. Thus the points O', B', C' are really the same points as the points O, B, C but on these reflected 'images' of the playground. The shortest route from O to O' is certainly the straight line joining them, and this corresponds to the square route $OABC$ touching the four midpoints of the sides of the square.

The length of this route is four times the length of AB which is, by Pythagoras's theorem, $4\sqrt{15^2 + 15^2} = 84.85$ m to two decimal places. If Peter starts from any other point P on the side of the playground then he can take the rectangular route shown, which is the same length as the line PP' using the reflections of the playground. Since $POO'P'$ is a parallelogram the length PP' of Peter's route is the same as the length OO' of Cynthia's route, so Peter has no grounds for complaint.

87-6. ROUND TRIP?

In the left-hand figure, OHF is an equilateral triangle because of the angle 60° at O and the equal sides OH, OF. Thus $HF = 100$ m. It follows that, if John swims along HF at $\frac{1}{2}$ m/sec, then he will take 200 seconds to reach Fred's house. As this is his maximum swimming speed it will in general take at least 200 seconds to complete the trip. The arc distance HF round the edge of the lake is one-sixth of the circumference (since $60 = \frac{1}{6}360$), that is $\frac{1}{6} \times 2\pi \times 100 = \frac{100}{3}\pi$ m. To cover this distance in 200 seconds, John would have to walk at a speed of $100\pi/(3 \times 200) = \frac{1}{6}\pi$ m/sec, or 0.524 m/sec to three decimal places.

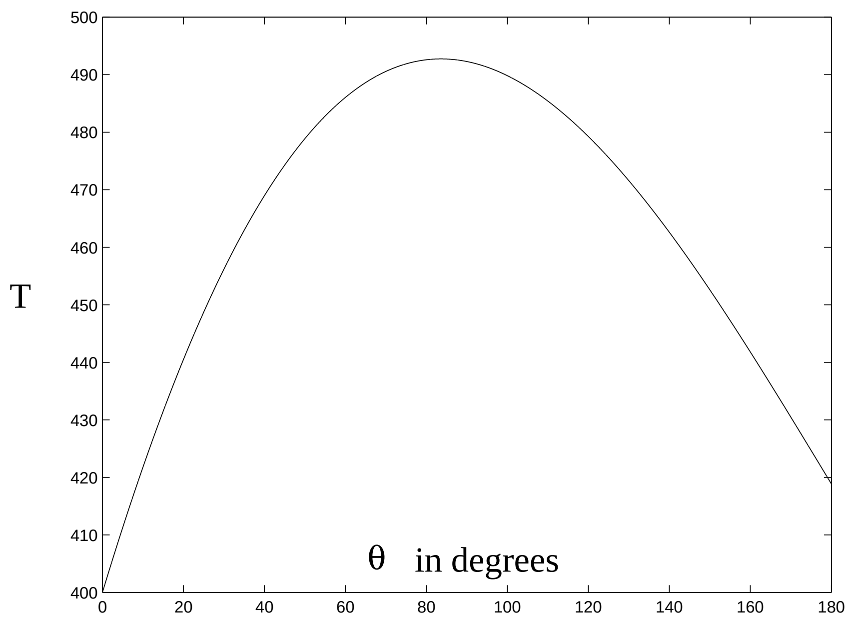




For the rest of the question we really need some trigonometry to write down the length of time John takes to walk and swim the route JPH drawn heavily in the right-hand figure. Thus the arc JP is $\frac{\theta}{360} \times$ circumference, that is $\frac{5}{9}\pi\theta$ m, provided θ is measured in degrees. As John walks $1\frac{1}{2}$ times as fast as he can swim, his walking speed is $\frac{3}{4}$ m/sec. Hence the time taken for him to walk from J to P round the edge of the lake is $\text{arc}JP/\frac{3}{4} = \frac{20}{27}\pi\theta$ sec.

From geometry, the angles OHP and OPH are both equal to $\frac{1}{2}\theta$, and $PH = 2\times 100\cos(\frac{1}{2}\theta) = 200\cos(\frac{1}{2}\theta)$ m. Therefore the time taken to swim across PH is $PH/\frac{1}{2} = 400\cos(\frac{1}{2}\theta)$ sec.

The total time for the route JPH is therefore $T(\theta) = \frac{20}{27}\pi\theta + 400\cos(\frac{1}{2}\theta)$ sec. Plotting this graph from $\theta = 0$ (that is $P = J$, swimming all the way) to $\theta = 180$ (that is $P = H$, walking all the way) gives the shape shown in the figure. It is then clear that the smallest value occurs at



the left-hand end, that is $\theta = 0$, so John's fastest route is to swim directly across the lake from J to H at his maximum swimming speed.

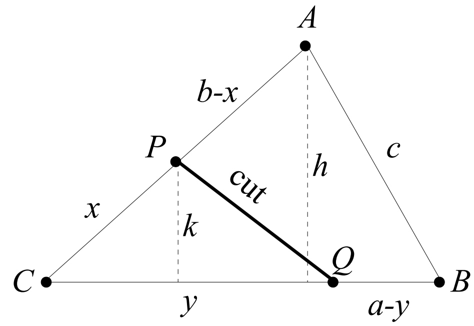
[The route that takes *longest*, from the graph, is for about $\theta = 83°$. Using methods of calculus, it can be shown that the maximum value of T occurs when $\sin(\frac{1}{2}\theta) = \frac{2}{3}$, and this gives $\theta = 83.62°$ to two decimal places. Notice also that the value of T for $\theta = 180°$ is only slightly more than the smallest value of 400. In fact it is about 418.9 sec.]

Senior Challenge 1988

88-1. FOOL'S GOLD

Let N be the minimum number of gold pieces in the hoard. Then $N - 1$ must be a multiple of *both* 20 and 30, i.e. a multiple of 60. Hence N must be one of the numbers $61, 121, 181, 241, \ldots$. But the question implies that N is divisible by 11. The smallest of the above numbers which satisfies this condition is 121, so this is the value of N.

88-2. PIECE OF CAKE



Our triangular slab of cake is of uniform thickness (1 inch), so if Kim and I are to receive equal helpings, each piece must have the same area on top. As the knife could be inserted into any of the three edges, let us be general and make the edges $AB = c$, $BC = a$, $CA = b$ inches as shown in the figure.
The area of the triangle ABC is $\frac{1}{2}ah$ where h is the perpendicular height shown. Letting the cut be along PQ, where $CP = x$, $CQ = y$, the area of CPQ is $\frac{1}{2}yk$ where k is the perpendicular height of the triangle CPQ. But by similar triangles we have $h/k = AC/PC = b/x$ so that the area of CPQ is in fact $\frac{1}{2}yk = \frac{1}{2}yhx/b$. If this is to equal half the area of ABC then we need $xy = \frac{1}{2}ab$. This is our first equation.

Also Kim and I are each to get equal helpings of chocolate: this means that the lengths of the sides covered in chocolate must be equal: $PC + CQ = PA + AB + BQ$, that is $x + y = b - x + c + a - y$, that is $x + y = \frac{1}{2}(a + b + c)$.

In the actual cake given in the problem, $a + b + c = 7 + 8 + 9 = 24$, so the last equation gives $x + y = 12$. Substituting $y = 12 - x$ in $xy = \frac{1}{2}ab$ we get

$$x(12 - x) = \tfrac{1}{2}ab, \quad \text{that is} \quad x^2 - 12x + \tfrac{1}{2}ab = 0.$$

This can be solved for x in terms of a, b by the quadratic formula or by completing the square: $(x - 6)^2 = 36 - \frac{1}{2}ab$, so that

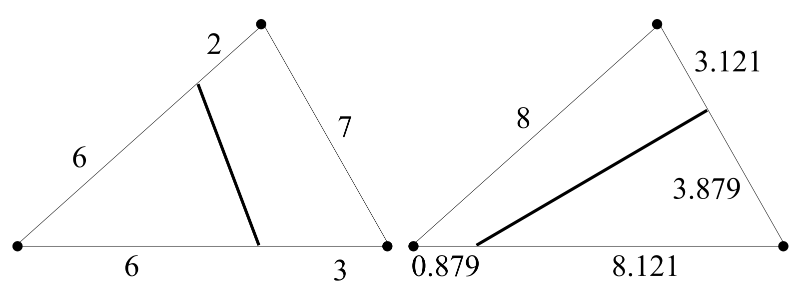
$$x = 6 \pm \sqrt{36 - \tfrac{1}{2}ab}. \qquad (1)$$

Case 1 I cut from the 8 inch edge to the 9 inch edge: $a = 9, b = 8$ so that $\frac{1}{2}ab = 36$ and the last formula gives $x = 6$ and $x + y = 12$ gives $y = 6$.

Case 2 I cut from the 8 inch edge to the 7 inch edge: $a = 7, b = 8$ so that $\frac{1}{2}ab = 28$ and equation (1) gives two solutions, $x = 8.828, 3.172$ to 3 decimal places. But the first solution is $> b$ so is obviously no use (I should have to start cutting in thin air outside the slab!). Also using $x + y = 12$ the other solution for x gives $y = 8.828 > a$ so neither of these is a physically significant solution.

Case 3 I cut from the 7 inch edge to the 9 inch edge: $a = 9, b = 7$ so that $\frac{1}{2}ab = 31.5$. Using equation (1) we find (again to 3 decimal places) $x = 8.121 > b$, so rejected as before, and $x = 3.879, y = 8.121$.

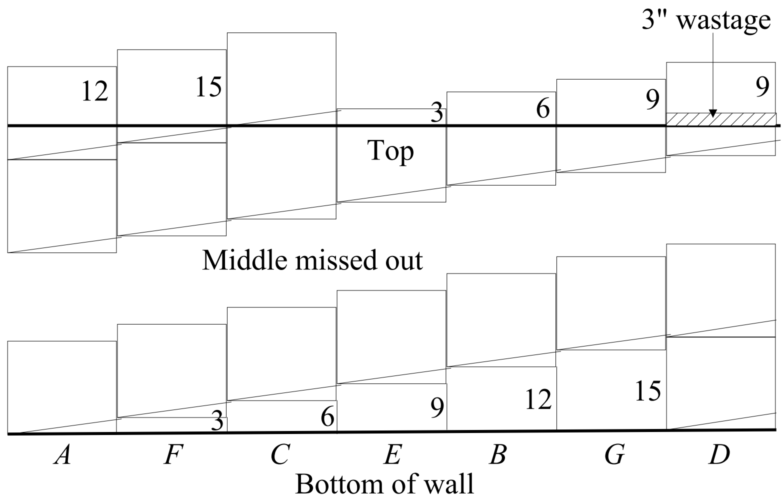
There are thus two solutions to the problem, sketched in the figure.



88-3. THE HASTY PASTER

The wall is 147 inches long and the paper is 21 inches wide, so 7 vertical strips are needed to cover the wall. The wall is 96 inches high and each pattern block is 18 inches long so 5 complete blocks + 6 inches of the next block will fit up the wall. The pattern rises by 3 inches across each strip; hence if the strips are fitted progressively left to right, Wendy would have to waste $12 - 3 = 9$ inches of each of the first six strips to make the patterns match, a total waste of 54 inches !

The figure shows how this waste can be drastically cut to 3 inches. Each pattern block is represented by a rectangle, 21 across by 18 high, with a diagonal line from bottom left to a distance of 3 up the right-hand edge. (The diagram is approximately to scale.) The pattern



would be continued above the top of the wall as shown. Four strips, A, B, C, D each 96 inches long are cut from the roll. Then 3 inches of wastage is cut from D. Three more strips E, F, G are cut from the roll and the seven strips are pasted on the wall in the order shown. The total length needed is $7 \times 96 + 3 = 675$ inches of wallpaper.

88-4. CHEW IT OVER

The figure shows Farmer Fermat's[7] rectangular garden with Clarence at C. Let x, p, q, r, s be as in the diagram, where PQ is drawn parallel to the sides AE, BD and hence perpendicular to the sides AB, ED. Then, applying Pythagoras's theorem:

$$\Delta ACP: \quad p^2 + r^2 = 25; \quad \Delta BCP: \quad p^2 + s^2 = 100; \text{ subtracting, } s^2 - r^2 = 75,$$

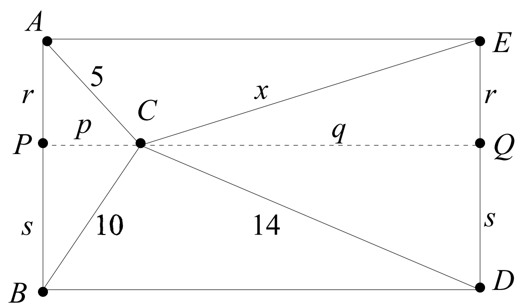
$$\Delta DCQ: \quad q^2 + s^2 = 196; \quad \Delta ECQ: \quad q^2 + r^2 = x^2; \text{ subtracting, } s^2 - r^2 = 196 - x^2.$$

Equating these two gives $x^2 = 121$, that is $x = 11$. Thus Clarence is 11 m from E.

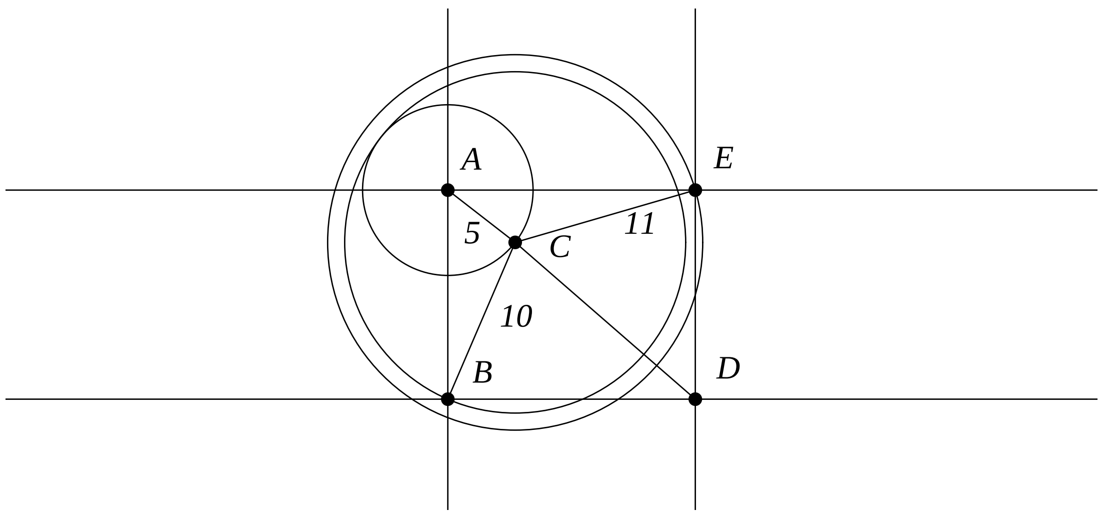
Interestingly, we cannot solve for the other unknowns in the equations, and indeed the shape of the rectangular garden is not determined by the given information. In the figure[8] two

[7] The reference is to the great French mathematician Pierre de Fermat (1601-65) who, besides being the author of the most famous 'last theorem' in number theory [9], worked a great deal in geometry and the precursors of calculus. The Fermat point of a triangle is the point the sum of whose distances from the three vertices is as small as possible. Hence the reference! See [10, p. 75], [3, p.22]; you can find much material on Fermat in [5]. The Fermat point has also been called the Steiner point and the Torricelli point.

[8] This figure was produced using *Cinderella* [2] which enables the figure to be animated, with C moving round the circle centre A and the points B, D, E moving accordingly.



perpendicular lines are drawn through A, and a circle of radius 5 is drawn centred at A. Then *any point* C is chosen on this circle, below the horizontal line through A and to the right of the vertical line through A. With centre at C a circle radius 10 cuts the vertical line; call this



point B. With centre C a circle radius 11 cuts the horizontal line; call this point E. Completing the rectangle gives the point D and the distance CD will be 14 automatically. Note that in particular C could lie on the vertical line or the horizontal line through A !

88-5. GENERAL SOLUTION

General Issimo shrewdly observes that the enemy will count the number of troops in a *corner* position *twice* when totalling the numbers of each of the adjacent sides. Hence, as casualties occur, he will move his men progressively to the corner positions keeping 15 on each side. There are many ways of doing this, and the figure on the next page shows one of them. Note that 30* indicates that the General himself takes the place of the eleventh man to fall. The procedure illustrated retains an approximately equal number of troops in each corner to ensure that the fort is adequately defended on all sides. With 15 on each side, the last tenable position must be with 30 active defenders, including Issimo himself of course! When reinforcements arrive, the enemy are about to charge, so there must be only 29 active defenders remaining.

88-6. TIGHT CORNER!

We give below a geometrical solution; of course it is also possible to investigate this problem 'experimentally' by means of accurate models or diagrams.

40

5	5	5
5		5
5	5	5

39

6	4	5
4		5
5	5	5

38

6	4	5
4		4
5	4	6

37

6	3	6
4		3
5	4	6

36

6	3	6
3		3
6	3	6

35

7	2	6
2		3
6	3	6

34

7	2	6
2		2
6	2	7

33

7	1	7
2		1
6	2	7

32

7	1	7
1		1
7	1	7

31

8		7
		1
7	1	7

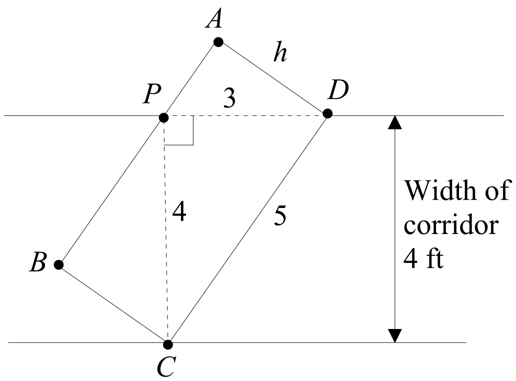
30

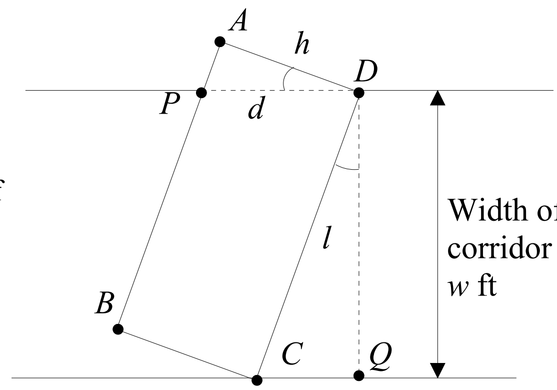
8		7
7		8

30*

8		7
7		8

In the left-hand figure, PD is the doorway of width 3 ft, and $ABCD$ is the *largest* rectangular trunk which Bill and Ben can manoeuvre through the doorway without tipping. It has length 5





ft and width h ft as shown.

Now triangle DPC has sides 3, 4, 5 and hence by Pythagoras's theorem is right-angled at P. The area of this triangle can be found in two ways, using the formula $\frac{1}{2}$ base $\times$ height:

$$\text{Base } PC,\ \text{height } PD: \ \text{area } = \tfrac{1}{2} \times 4 \times 3 = 6 \text{ sq ft},$$

$$\text{Base } CD,\ \text{height } AD: \ \text{area } = \tfrac{1}{2} \times 5 \times h = 5h/2 \text{ sq ft}.$$

Equating these, we deduce $h = \frac{12}{5}$.

Hence the trunk of width 2 ft *will* go through the doorway as $2 < h$, whereas the trunk of width $2\frac{1}{2}$ ft will *not* as $2\frac{1}{2} > h$. The area of the top of the largest trunk is $5h = 12$ sq ft.

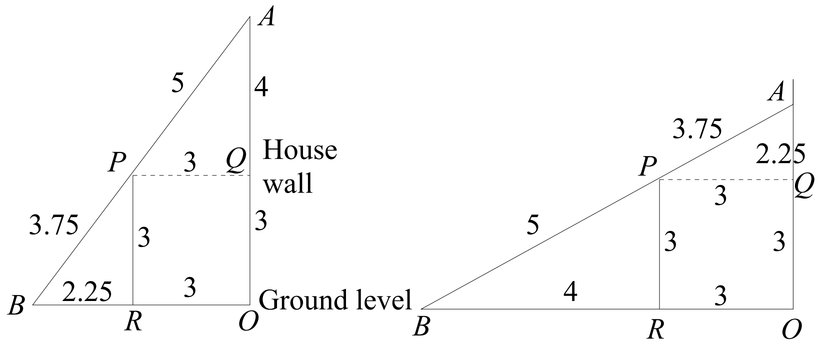
To give a complete answer we need to consider the case where the length of the trunk is not equal to 5 ft, but the corridor is still 4 ft wide and the door is still 3 ft wide. For the present we shall use general letters for all the distances involved. In the right-hand figure, the width of the corridor is w, that of the doorway d, and the length of the trunk is l. A perpendicular has been dropped from D to the far side of the corridor, meeting it at Q. Since ADC and PDQ are both right-angles, subtracting angle PDC from them shows that the marked angles ADP and QDC are equal. In fact, the triangles ADP and QDC have the same angles (one of them a right-angle), and so are *similar*. This means that the ratios of corresponding sides are equal, which means that $h/d = w/l$, that is $hl = wd$. The area hl of the face of the largest trunk which will go through the doorway is therefore equal to wd, the product of the width of the corridor and the width of the doorway. In the case $w = 4, d = 3$ this is 12 sq ft, and the answer did not after all depend on the special choice of 5 for the length of the trunk.

Senior Challenge 1989

89-1. OPENERS

				Fold			
These	$\rightarrow$	1	2	|	$n-1$	n	Back page = page n
are	$\rightarrow$	3	4	|	$n-3$	$n-2$	
the	$\rightarrow$	5	6	|	$n-5$	$n-4$	
		$\vdots$	$\vdots$	|	$\vdots$	$\vdots$	
double	$\rightarrow$	25	26	|	$n-25$	$n-24$	Given as page 46
		$\vdots$	$\vdots$	|	$\vdots$	$\vdots$	Therefore $n-24=46$,
sheets	$\rightarrow$	33	34	|	37	38	so $n=70$.
Single sheet	$\rightarrow$		35	|	36		

89-2. OVER THE TOP



In the left-hand figure, PR is the small wall running parallel to the house wall. When the ladder reaches its maximum height up the house wall, it rests against the small wall at P. Draw PQ perpendicular to AO. Then $PQ = 3$ metres and $PQOR$ forms a square of side length 3 metres. Hence $AQ = 4$ metres. Applying Pythagoras's theorem to the right-angled triangle APQ we get $AP^2 = AQ^2 + PQ^2 = 16 + 9 = 25$ so $AP = 5$ metres. Now triangles APQ and PBR are *similar* as corresponding angles are equal: in fact triangle BPR is just a copy of triangle APQ scaled down by a factor of $\frac{3}{4}$. Hence $PB = \frac{3}{4}AP = 3.75$ metres. The ladder is therefore 8.75 metres long. Also $BR = \frac{3}{4}OR = 2.25$ metres.

When the ladder is moved down the house wall, it will lose contact with the top of the small wall until A is at its minimum height,, when it will again rest on the small wall at P, as in the right-hand figure. The position of the ladder will then be a mirror image (in line OP) of its former position. Hence $AQ = 2.25$ metres and the height reached up the house wall will be 5.25 metres.

89-3. PIZZA PIE

The answer to this problem depends crucially on whether or not portions of pizza can be cut again after purchase!

Pizza diameter (inches)	8	12	16
area (square inches)	16π	36π	64π
cost (pounds)	2	3	4

Since all pizzas have the same uniform thickness, the quantity received will be proportional to area. Then
Let S be the area of $\frac{1}{4}$ portion of small pizza (4π sq.ins.), costing 50p,
let M be the area of $\frac{1}{4}$ portion of medium pizza (9π sq.ins.), costing 75p, and
let L be the area of $\frac{1}{4}$ portion of large pizza (16π sq.ins), costing £1.

Solution A This solution assumes that pizzas *can* be further divided after purchase.
Cost per unit area: Small pizza £$\frac{1}{8\pi}$ > Medium pizza £$\frac{1}{12\pi}$ > Large pizza £$\frac{1}{16\pi}$.
Hence the best plan is to buy as much of the large pizza as possible and divide the lot into 14 equal servings. The total required is $14 \times 36\pi = 504\pi$ sq. ins $= 31L + 8\pi$ sq.ins.
Hence for minimum cost: exact solution is $31L + 2S$: $\frac{1}{2}$ of small pizza + $7\frac{3}{4}$ of large pizzas, total cost £32.

However, if *wastage* is allowed, then $31L + M$ gives an extra π sq.ins. which can be shared out! Then $\frac{1}{4}$ medium pizza + $7\frac{3}{4}$ large pizza give a total cost of £31.75.

Solution B This solution assumes that pizzas *cannot* be cut up again. Note that it is not obvious that each person receives the same combination of S, M and L portions; we need to prove this to be true. We know that each person must receive a serving of 36π sq.ins. Hence the possible servings are $9S, 4M, 5S+L$ or $S+2L$. Suppose that x people receive $9S$, y receive $4M$, z receive $5S + L$ and w receive $S + 2L$, where of course

$$x + y + z + w = 14. \tag{2}$$

Then

$$\begin{aligned}\text{Total quantity purchased} &= 9Sx + 4My + (5S + L)z + (S + 2L)w \text{ square inches} \\ &= (9x + 5z + w)S + 4yM + (x + 2w)L \text{ square inches}\end{aligned}$$

The total cost is (in pounds)

$$\begin{aligned}C &= (9x + 5z + w) \times \tfrac{1}{2} + 4y \times \tfrac{3}{4} + (z + 2w) \times 1 \\ &= \tfrac{1}{2}(9x + 6y + 7z + 5w)\end{aligned}$$

But from (2) we have $5w = 70 - 5x - 5y - 5z$, so $C = \frac{1}{2}(70 + 4x + y + 2z)$, which is clearly a minimum when $x = y = z = 0$, which makes $w = 14$. So everyone *does* in fact receive the same, namely $S + 2L$. The total cost is £35.

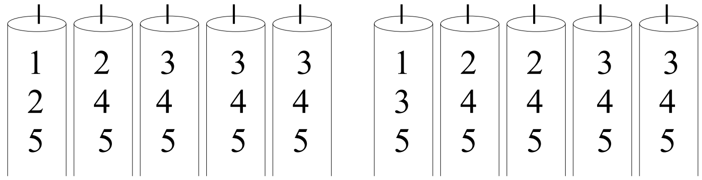
89-4. A BURNING QUESTION

Each candle alight burns down 1 cm. during Evensong.

During Advent 4 candles burn down by a total amount $1+2+3+4 = 10$ cm. before Christmas. However, as 10 is not a multiple of 4, it is *not* possible for each candle to burn down by the same whole number of centimetres before Christmas.

During Lent This time the total burn-down is $1 + 2 + 3 + 4 + 5 = 15$ cm. Hence solutions *are* possible whereby each candle burns down 3cm. during the 5 Sundays before Easter.

Two possible solutions are shown in the figure on the next page in which burn-down during successive Sundays is indicated by 1, 2, 3, 4 and 5.

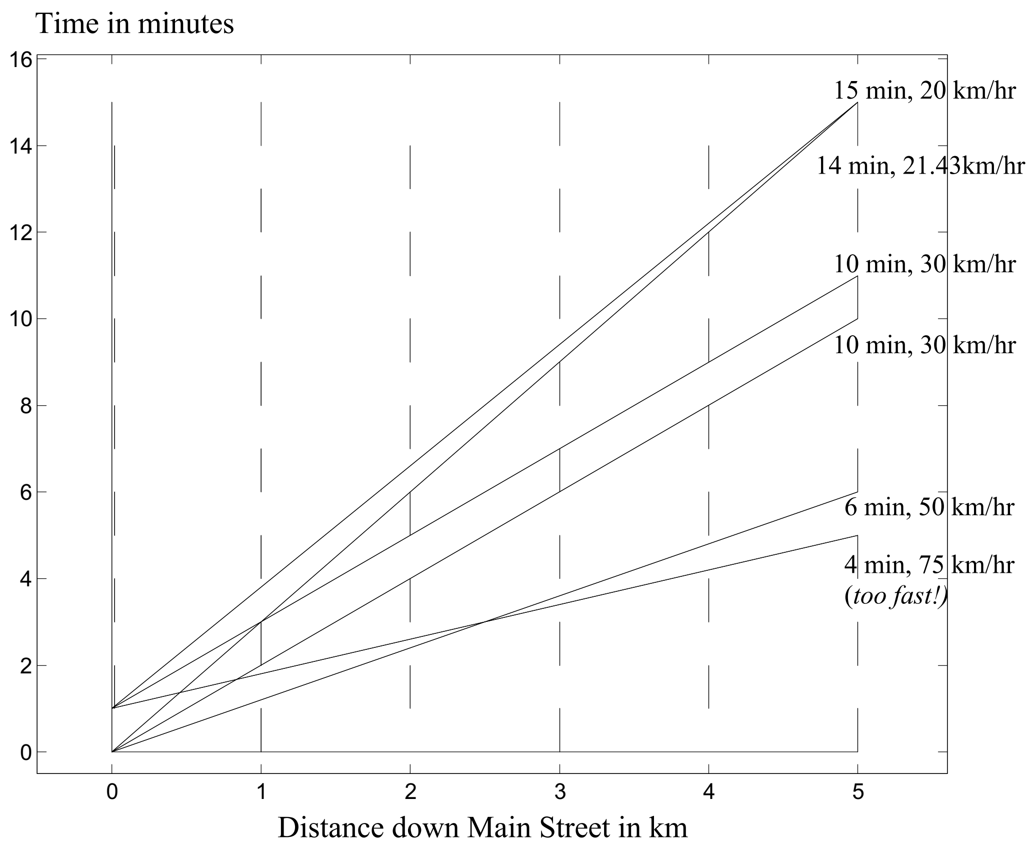


89-5. GREEN LIGHT

In the diagram, a red light on a vertical line is given by | and a green light by a blank. So the object is to draw a straight line from some point between $(0,0)$ and $(0,1)$ which avoids all the 'obstacles' and whose slope represents a speed of ≤ 70 km/hr. If Herbie travels at a constant speed of v km/hr, ranges of possible speeds as given by the diagram are:

$20 < v \leq 21.43$, $v = 30$ if he takes very little time to get through the lights!, $50 < v < 70$.

Note that 20 km/hr could also be achieved by a straight line joining $(0,1)$ to $(5,16)$, a time of 15 minutes.

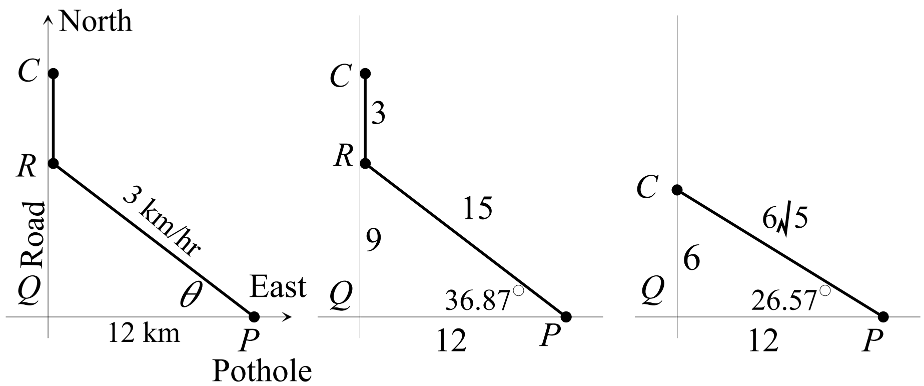


89-6. SALLY FORTH

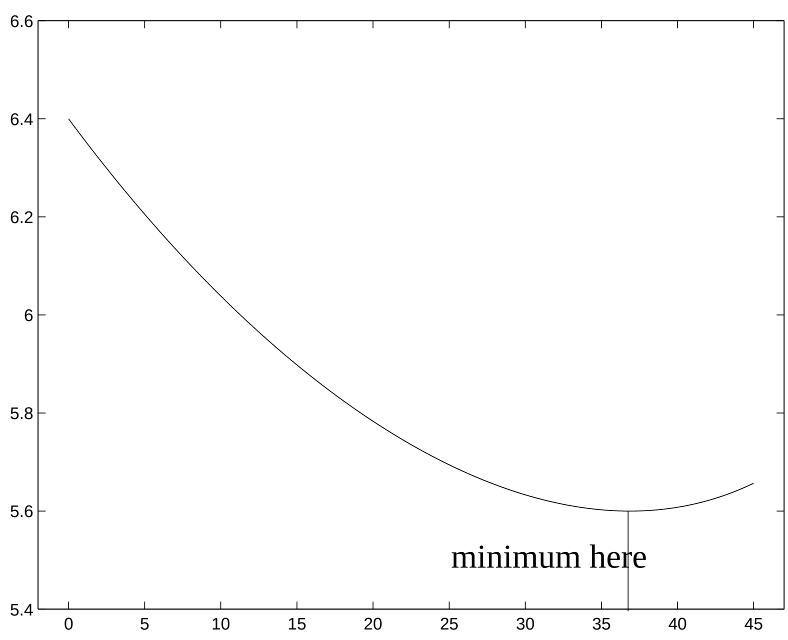
Referring to the left-hand figure, assume Sarah reaches the road at R, so that $PR\cos\theta = 12$, i.e. $PR = 12/\cos\theta$, and $RQ/12 = \tan\theta$, i.e. $RQ = 12\tan\theta$. Hence the total time T in hours for Sarah to travel from P to C is

$$\frac{PR}{3} + \frac{RC}{5} = \frac{4}{\cos\theta} + \frac{x - 12\tan\theta}{5},$$

where $x = QC$. This formula is invalid if R is further north up the road than C. In this case $RC = x + RQ$. But we do not have to consider this case as the time taken by Sarah to go from P to C via R would always be greater than the time she would take if she went *directly* across the moor from P to C!



Case 1: $QC = x = 12$ km. Then $T = \frac{4}{\cos\theta} + \frac{12}{5}(1 - \tan\theta)$. Clearly the angle θ must be $< 45°$ in this case for R to be south of C. By calculating values of T, or by plotting a graph, or indeed if you know how to do it by calculus, you find that the smallest value of T occurs when θ is about $37°$, and then $T = 5.6$ hours to 1 decimal place. See the graph and the middle figure. (The exact figure is $\sin\theta = \frac{3}{5}$, giving $\theta = 36.87°$ to 2 decimal places.)



Case 2: $QC = x = 6$ km; see the right-hand figure. Here, $T = \frac{4}{\cos\theta} + \frac{12}{5}(1 - \tan\theta) - \frac{6}{5}$. This is just the previous T shifted down by 1.2, so the shape of the graph is the same, with a minimum

at about $\theta = 37°$, but it is only valid for $0 < \theta <$ the angle whose tangent is $\frac{6}{12} = \frac{1}{2}$, i.e. (working to 2 decimal places from now on) $0 < \theta < 26.57°$. For larger θ the point R is north of C and so we conclude that the quickest route for Sarah to take is with $\theta = 26.57°$ i.e. directly across the moor from P to C. The time taken is $6\sqrt{5}/3 = 4.47$ hours.

Senior Challenge 1990

90-1. No. PROBLEM!

We know

$$\begin{aligned} abcdef \times 2 &= cdefab & (3)\\ abcdef \times 3 &= bcdefa & (4)\\ abcdef \times 4 &= efabcd & (5)\\ abcdef \times 5 &= fabcde & (6)\end{aligned}$$

If $a \geq 2$, (6) would have 7 digits; hence $a = 1$.
Hence (4) ends in 1; so $f \times 3$ is a number ending in 1; hence $f = 7$.
As $f \times 2 = 14$, (3) must end in 4, so $b = 4$.
Similarly from (5), $d = 8$ and from (6) $e = 5$.
Now (5) states: $14c8\underline{57} \times 4 = 5714\underline{c8}$. As $57 \times 4 = 228$, $c = 2$.
Hence $abcdef = 142857$. When multiplied by 6 this gives 857142, which is $defabc$.
On the other hand, $142857 \times 7 = 999999$ which is quite different!

Note that the digits in $2 \times 142857, \ldots, 6 \times 142857$ are all *cyclic* permutations of 142857: writing the six digits round a circle the digits in each of these goes once round the circle in the same direction. But all the cyclic permutations of 142857 have now been used up so *something* different has to happen!

The explanation really lies in the fact that $\frac{1}{7} = 0.142857$ recurring, that is the block of digits 142857 repeats forever. The fractions $\frac{2}{7}, \ldots, \frac{6}{7}$ consist of recurring blocks of digits exactly as above. But $\frac{7}{7}$ is certain to be different since it equals 1, which is the same as 0.9 recurring.

90-2. A SWITCH IN TIME

On Tim's clock, the minute hand M_T has been fitted onto the hour hand spindle and the hour hand H_T has been fitted onto the minute stand spindle.

Obviously this will not matter if M_T and H_T are coincident. However, when Tim collects his clock, this may not be so as M_T and H_T could have just been *set* at the correct time! It is better to compare the motion of M_T with that of the minute hand (M) on a normal clock (or H_T with the motion of the hour hand H on a normal clock).

When Tim's clock shows the correct time, M and M_T will coincide. During the next 12 hours M will pass M_T *ten* times and then approach M_T as the 12-hour period draws to an end. Hence in any 12 hour period, M and M_T will coincide *eleven* times. Therefore, Tim's clock will show the correct time every $\frac{12}{11}$ hours, that is at intervals of 1 hour 5 minutes 27.27 seconds.

90-3. TUNNEL VISION

Fast cars travel at 55,000 metres per hour. The gap between them is 25 metres.
Slow cars travel at 35,000 metres per hour. The gap between them is 20 metres.

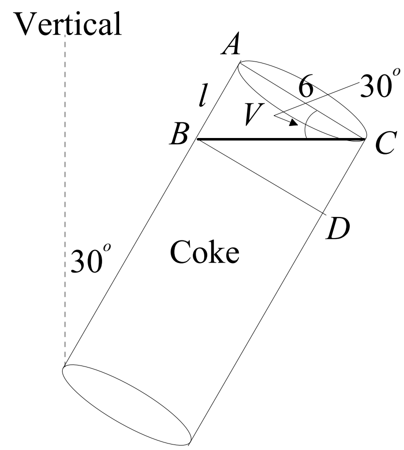
If my brother and I travel at u metres per hour and cars coming the other way are moving at v metres per hour, then they *appear* to us to be passing with a speed of $u + v$ metres per hour. Hence if they are d metres apart, they pass us at the rate of $\frac{u+v}{d}$ cars per hour.

Hence if we are in the fast lane,
I will count $(55,000 + 55,000)/25 = 4400$ fast cars per hour.
my brother will count $(55,000 + 35,000)/20 = 4500$ cars per hour.
So he counts more cars.

However if we are in the slow lane
I will count $(35,000 + 55,000)/25 = 3600$ fast cars per hour.
My brother will count $(35,000 + 35,000)/20 = 3500$ cars per hour.
So I count more cars.

(Of course the drive through the tunnel takes much less than an hour at these speeds! But the comparison of numbers of cars remains valid for the shorter length of time: the actual numbers counted will be the above numbers multiplied by the same factor in each case, namely the time as a fraction of an hour taken to drive through the tunnel.)

90-4. HIGH AND DRY



When Angela's glass tilts, the surface BC of the fluid inside just reaches the rim at C as shown. Let $AB = l$. Then ABC is right-angled with angle $ACB = 30°$. Thus as the diameter of the glass AC is 6 cm, we have $l = 6\tan 30° = 6/\sqrt{3} = 3.46$ cm, giving results to 2 decimal places. The cross-sectional area G of the glass is 9π sq.cm. The volume V of coke which Angela drank before the glass tilted is half the cylindrical volume $ABDC$ which equals $\frac{1}{2}lG = 9\pi\sqrt{3} = 48.97$ cu.cm.

Note The above calculation is incorrect if the surface of the liquid in the tilted glass encroaches on the base. However we are given that the total volume of the glass is 330 cu.cm. So its length is $330/G = 11.67$ cm., which is $> l$ so B is well above the base.

90-5. NIL RETURN

Consider the product $1 \times 2 \times 3 \times \ldots \times 100$. How many factors of 5 are there in this? Every fifth number has a factor 5; this accounts for the numbers 5, 10, 15, 20, up to 100, that is 20 numbers. But certain of these have a factor 5^2, namely 25, 50, 75, 100. So each of these provides an *additional* factor of 5, making 24 factors of 5 in all. (That is, $100! = 5^{24}N$ where N is not a multiple of 5.)

The way to make zeros at the end of the product giving 100! is to combine factors of 5 with factors of 2. There are many more than 24 factors of 2 in the product (the even numbers 2, 4, 6, up to 100 alone provide 50 of them) so we run out of factors of 5 long before we run out of factors of 2. So the number of zeros at the end of 100! is exactly 24.

(A similar exercise for say 200! would have to take into account the fact that $125 = 5^3$ is among the numbers being multiplied. We would get in this case 40 multiples of 5, plus 8 multiples of 5^2, plus 1 multiple of 5^3, making 49 factors of 5, and hence 49 zeros at the end of 200!.)

It is much harder to get hold of the last nonzero digit (just before the 24 zeros) in 100!. Let us start with factorials up to 10!. We are greatly helped by the following rule:

> *to find the units digit of $a \times b$, take the units digit of a times the units digit of b and find the units digit of that.*

For example the units digit of 273×442 is $3 \times 2 = 6$ while the units digit of 273×448 is the units digit of $3 \times 8 = 24$, that is 4.

We need to apply this rule repeatedly, but also to make allowance every time an extra zero appears at the end of our factorial, for we want to ignore the powers of 10 and find not the last digit but the last *nonzero* digit of the factorial.

Now $3! = 6$ with last (nonzero) digit 6. Let's take the next two numbers, 4 and 5, together, since this will introduce a zero. We want to ignore the 10 so instead of multiplying by 4 and 5 we multiply by 2 and 1, taking a factor 2 out of the 4 leaving 2, and a factor 5 out of the 5 leaving 1. So 5! has last nonzero digit given by 6 (from 3!) times 2 (from 4 and 5), that is the units digit of 6×2, which is 2. (Of course, $5! = 120$ so this is correct!)

For 6! up to 9! we multiply by 6, 7, 8 and 9 in turn, all the time considering the units digit only:
2×6 gives 2: last nonzero digit of 6!
2×7 gives 4: last nonzero digit of 7!
4×8 gives 2: last nonzero digit of 8!
2×9 gives 8: last nonzero digit of 9!
and of course multiplying by 10 to get 10! leaves the last *nonzero* digit the same: 8.

We can apply the same method going from 10 to 20, then 20 to 30 and so on. Consider multiplying by 11, then 12, up to 20. Proceeding as before,
8×1 gives 8: last nonzero digit of 11!
8×2 gives 6: last nonzero digit of 12!
6×3 gives 8: last nonzero digit of 13!
Now take 14 and 15 together, taking a factor 2 out of 14 to give 7 and a factor 5 out of 15 to give 3: 8×7 gives 6, then 6×3 gives 8: last nonzero digit of 15!
8×6 gives 8: last nonzero digit of 16!
8×7 gives 6: last nonzero digit of 17!
6×8 gives 8: last nonzero digit of 18!
8×9 gives 2: last nonzero digit of 19!
Now we introduce an extra factor of 10: 2×2 gives 4: last nonzero digit of 20!

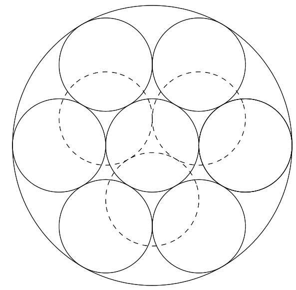
Continuing in this way we can work our way up to 100!, making allowances every time there is an extra zero, or occasionally an extra two zeros, about to appear in the factorial. For example between 21 and 30 we encounter 25, which has two factors of 5, so we must compensate by taking two factor of 2, say both of them from the preceding number 24, leaving 6. In this way the last nonzero digit of 25! comes from $4 \times 2 \times 3 \times 6$, that is 4 is the last nonzero digit of 25! The following table shows the last nonzero digits of the $N!$ where N is a multiple of 5, up to 100!, worked out in exactly the above way.

N	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
Last $\neq 0$ digit of $N!$	8	2	2	6	2	4	6	6	8	4	8	2	2	6	4

Particular care is needed at 50!, where there is an extra factor of 5, so we take a factor of 2 out of a number close to 50, say 48, leaving 24. The 50 and this 2 make 100, which doesn't change the last nonzero digit. The same care is needed at 75!, where there are two extra factors of 5, so we take these out of 75, leaving 3, and take two factors of 2 out of numbers just less than 75. We could take both of them out of 72, leaving 72/4=18.

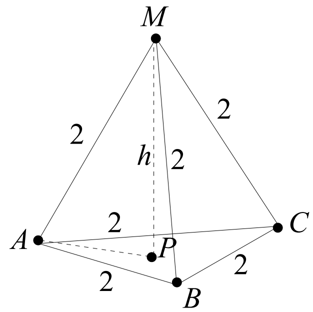
It is clear that it would not be an easy matter to give a formula for the last nonzero digit of $N!$ for a general value of N. In fact no general formula is known; it is not even known whether the sequence of last nonzero digits is 'eventually periodic', that is whether, from some point onwards, the same sequence of numbers repeats forever.

90-6. MARBLE GEOMETRY

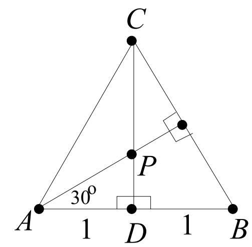


It is possible to place *seven* marbles on the bottom of the container. They form a hexagonal pattern touching each other and the sides of the container, as shown by the solid lines in the diagram. If a second layer of marbles is formed so that each one touches three others in the bottom layer it is possible to have only *three* marbles in this layer, as shown by the dashed lines in the figure.

As spheres always touch on a straight line joining their centres, the centres must be 2 cm. apart, since this is the diameter of the marbles. The centre M of a marble in the upper layer and those of the marbles in the bottom layer which touch M form the corners (vertices) of a regular *tetrahedron* of edge length 2 cm. This is shown in the diagram on the right.



MP is perpendicular to the base



Base of tetrahedron

Here A, B and C are the centres of the marbles in the bottom layer. Each face of the tetrahedron is an equilateral triangle, and P is the centre of the triangle ABC. We have $AP = AD/\cos 30° = 2/\sqrt{3}$ cm. The height $h = MP$ is obtained from the right-angled triangle APM: $h^2 = 4 - (2/\sqrt{3})^2 = 4 - \frac{4}{3} = \frac{8}{3}$, so $h = 2\sqrt{2}/\sqrt{3}$ cm.

The height of M above the base of the container equals $h+$ the height of the centres of the marbles with centres A, B, C above the base $= h + 1 = 2.63$ cm. to 2 decimal places.

90-7. COMMON CENSUS

The other three people living at 500 College Avenue have *different* ages. The youngest is at least 3 years old, and the product of the three ages is 900. The table shows all the possible combinations of ages with the *middle* ages in ascending order down the table.

Possible Ages			Sum of eldest and one of the other two	
3	4	75		
3	5	60	63	65
4	5	45	49	50
3	6	50	53	56
5	6	30	35	36
4	9	25	29	34
5	9	20	25	29
3	10	30	33	40
5	10	18	23	28
6	10	15	21	25
3	12	25	28	37
5	12	15	20	27
3	15	20		

Stage 1 The owner tells Chris the age of the *middle* person. Now if this had been 4 or 15, Chris

would have known the answer to the problem straight away, as the combinations corresponding to 4 and 15 are not repeated: hence we can reject $(3, 4, 75)$ and $(3, 15, 20)$.

Stage 2 The owner tells Polly the sum of the ages of the eldest and one of the other two. These sums are listed in the second column of the table, for the combinations not rejected in Stage 1, with repeated numbers boxed. As Polly cannot use this information to solve the problem immediately we can eliminate the combinations $(3, 5, 60), (4, 5, 45), (3, 6, 50), (5, 6, 30), (3, 10, 30)$ and $(5, 12, 15)$ which have sums not repeated elsewhere.

Stage 3 We can also reject $(3, 12, 25)$ now as the only other one with 12 as the middle age has been discounted in Stage 2. This means the Polly's sum could not be 28—hence $(5, 10, 18)$ can be eliminated as well! The only combination left with 10 in the middle is $(6, 10, 15)$ so this has to go too, otherwise Chris would have chosen it.

Stage 4 It is now clear that Chris was told that the middle age is 9 and cannot choose between the combinations $(4, 9, 25)$ and $(5, 9, 20)$. Polly would be stuck on the horns of the same dilemma if she had been told 29. However she is not so stuck. Hence her sum must have been 25 and the correct combination is 5, 9 and 20.

Senior Challenge 1991

91-1. CLARIFICATION

The magic multiplier for wine is $1.20 \times 1.05 \times 1.15 \times 0.9 = 1.3041$.
The magic multiplier for spirits is $1.25 \times 1.05 \times 1.15 \times 0.85 = 1.2830$.

Note The answer obtained by dividing £3.91 by £3.00, namely 1.3033, is not accurate enough for determining the shelf price of bottles of wine costing more than £7. This is because £3.91 has been rounded.

91-2. PEDIGREE CHUMS

Sue makes three statements:
(A) If I have a sheepdog, but not a terrier, I also have a poodle.
(B) I *either* have both a poodle and a terrier *or* neither.
(C) If I have a poodle, then I also have a sheepdog.

As we are told that Sue has *at least one* dog at home, her second statement (B) leaves us with only three possibilities:

(i) poodle + terrier, *or* (ii) sheepdog only, *or* (iii) all three breeds.

Statement (A) excludes (ii), and statement (C) excludes (i). Hence we conclude that Sue has *all three breeds* of dog at home.

91-3. QUARTERED

Let FOURTH be expressed as $(\text{F}\times 100,000) + N$, where N is the 5-digit number OURTH. Then OURTHF = $(N \times 10)$+F. But FOURTH is given to be 4× OURTHF, that is 100,000 F + N = $40N$ + 4F, which when rearranged gives $N = 2564$F.

It follows that FOURTH =100,000 F + 2564 F = 102564 F. So FOURTH can be *any* single-digit multiple of 102564. These are:
102564, 205128, 307692, 410256, 512820, 615384, 717948, 820512, 923076.

91-4. COORDINATION

The highest common factors (= greatest common divisors) of the coordinates are shown in the table.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3	1
4	1	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4
5	1	1	1	1	5	1	1	1	1	5	1	1	1	1	5	1
6	1	2	3	2	1	6	1	2	3	2	1	6	1	2	3	2
7	1	1	1	1	1	1	7	1	1	1	1	1	1	7	1	1
8	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	8
9	1	1	3	1	1	3	1	1	9							

Here 1,...,9 at the left are the row coordinates and 1,...,16 on top are the column coordinates. Note that moves *up*, to the *left*, or *diagonally*, are not allowed!

(i) 9×9 board. The maximum score is obtained by keeping as close as possible to the diagonal. So move from top left to bottom right, alternately going one *down*, and one to the *right*. Total score is 1 + 1 + 2 + 1 + 3 + 1 + 4 + 1 + 5 +1 + 6 + 1 + 7 + 1 + 8 + 1 + 9 = 53.

(ii) 8×16 board. As the scores are larger towards the bottom of the board, the best strategy is to proceed as in (i) to square (8,8) and then move progressively to the right along the bottom. The total score is 43 (from the 8×8 array) + 1 + 2 + 1 + 4 + 1 + 2 + 1 + 8 = 63.

91-5. ESCAPE AID

For any cell to remain unlocked after the 100$^{\text{th}}$ warder had made his tour, it must have been visited by warders an *odd* number of times. Also, for cell number n to have been visited by warder number k, n must be a multiple of k (that is k is a factor of n, with $1 \leq k \leq n$).

Hence, if n has an even number of factors, including 1 and itself, cell number n was visited an even number of times and remains *locked* at the end. An example is $n = 48$: cell number n was visited by warders numbers 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48.

On the other hand, if n has an odd number of factors, cell number n received an odd number of visits and remained *unlocked* at the end. An example is $n = 36$, which was visited by warders 1, 2, 3, 4, 6, 9, 12, 18 and 36.

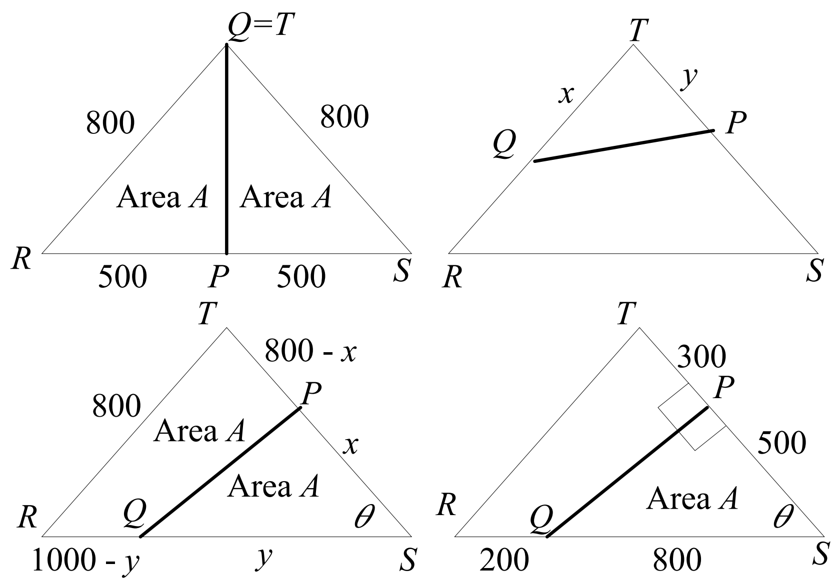
Now if n has an even number of factors, then they can be grouped in pairs which multiply to give n, for example $48 = 1 \times 48 = 2 \times 24 = 3 \times 16 = 4 \times 12 = 6 \times 8$. However, if n has an odd number of factors then the 'middle' factor has to be multiplied by itself to give n, for example $36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6$.

This means that if n has an odd number of factors (including 1 and n) then n must be a square number (a 'perfect square'). We conclude that the cells which were left unlocked at the end were numbered 1, 4, 9, 16, 25, 36, 49, 64, 81 and 100. Hence *ten* prisoners escaped when the warders had fallen asleep.

91-6. KNIGHT'S GAMBIT

Case 1 The most obvious way of dividing the triangular field so that each half has the same area A and the same perimeter is by letting the perimeter fence bisect the isosceles triangle as in the top left-hand figure. As angle QPR is 90 degrees we can use Pythagoras' theorem to find PQ: $PQ^2 = 800^2 - 500^2 = 390,000$, giving $PQ = 624.5$ metres approximately. Thus $A = \frac{1}{2} \times PR \times PQ = \frac{1}{2} \times 500 \times 624.5 = 156,125$ sq.m. approximately.

Case 2 See the bottom left figure.



Let the angle TSR be θ. Let $PS = x$ and $QS = y$ as shown. Then $A = \frac{1}{2}xy\sin\theta$ by a standard formula for the area of a triangle. The area of the whole triangular field is $\frac{1}{2} \times 800 \times 1000 \sin\theta = 2A$ so $xy = 400,000$.

The perimeter PSQ is $x + y + PQ$ and the perimeter of the quadrilateral $TPQR$ is $(800 - x) + PQ + (1000 - y) + 800 = 2600 + PQ - x - y$. Putting these two perimeters equal gives $x + y = 1300$ since PQ cancels out.

Combining the equations for xy and $x+y$ gives $400,000 = xy = x(1300-x) = 1300x - x^2$ so that $x^2 - 1300x + 400,000 = 0$, which factorizes into $(x - 500)(x - 800) = 0$ so that $x = 500$ or $x = 800$. Note that $x = 800$ gives the solution in Case 1. The new solution is $x = 500,\ y = 800$, as shown in the bottom right figure.

In fact, in the top left figure (Case 1), take the right-angled triangle QPS, turn it over and replace it on the big triangle as in the bottom right figure. This turns Case 1 into Case 2 and also shows that the angle at P is a right-angle in Case 2 !

Case 3 At first glance it might seem that you could also divide the field by letting the fence run from one 800 metre edge to the other, as shown in the top right figure. Following similar arguments to the above with $x = TQ$ and $y = TP$ you get $xy = 320,000$ and $x + y = 1300$, which by substituting for $y = 1300 - x$ in the first equation give $x^2 - 1300x + 320,000 = 0$. This equation needs to be solved by the 'quadratic formula' and gives x approximately 970 or 330 metres. Now $x = 970$ makes Q lie outside the triangular field and $x = 330$ makes $y = 970$ so P lies outside the field. So this won't work.

Senior Challenge 1992

92-1. SOFT CENTRED

Probably trial and error is as good as anything here, bearing in mind that giving one away from the full box must leave a multiple of 10 (since $\frac{9}{10}$ of the number of chocolates needs to be a whole number), so only numbers ending in 1 need be tested. You can get a bit nearer by going one more step: starting with $n = 10k + 1$ chocolates, after the first day there are $9k$ left, and then $9k - 2 = (10 - 1)k - 2 = 10k - (k + 2)$ must be a multiple of 10, which requires $k + 2$ to be a multiple of 10, say $k = 10l - 2$ and then $n = 100l - 19$. Trying our luck with $l = 1$, that is

$n = 81$, we find that it works! The numbers of chocolates on successive days are 72, 63, 54, 45, 36, 27, 18, 9, 0, so the chocolates lasted 9 days.

92-2. THE LIE OF THE LAND

Assume Sara tells the truth. Then Robert is not married and is a fisherman, which means that he lies, in which case Sara is not married and not a farmer, which implies that she is an unmarried fisherwoman, hence a lier. But this is a contradiction.

So in fact Sara lies, so we can deduce *Robert is a married farmer*, which means that he lies. Therefore *Sara is a fisherwoman who is not married*. This implies that she lies, and we have a consistent situation now.

92-3. FAST FOOD

Each revolution of Amy's wheel turns the centre wheel $\frac{52}{13} = 4$ times, so in one second the spinner turns $2 \times 4 = 8$ times. A piece of lettuce moves a distance π times the diameter, that is 25π cm in one revolution. In 8 revolutions the distance travelled is $8 \times 25\pi = 200\pi$ cm, and this takes place in 1 second. One hour is 60×60 seconds during which the lettuce travels $60 \times 60 \times 200\pi$ cm. Dividing by 100,000 which is the number of centimetres in a metre we arrive at about 22.62 km/hr.

92-4. GOBSMACKED

Firstly, any multiple of 3, apart from 3 itself, can be made up with small and medium packets alone. For $6 = S, 9 = M, 12 = 2S, 15 = S + M, 18 = 3S, 21 = 2S + M$ and so on. Furthermore *only* multiples of 3 can be made from S and M.

Consider next numbers which are one more than a multiple of 3, $N = 3r + 1$ say. Then $N - 40 = 3r - 39 = 3(r - 13)$ which as we know can be made up from S and M so long as $r \geq 15$. Since $40 = 2L$ (2 large bags) we can make up N from $2L$ and suitable S and M. However $r = 14$ gives $N = 43$ which cannot be made up since $43 - 20$ is not a multiple of 3 and $43 - 40 = 3$ which cannot be made from S and M.

When N is 2 more than a multiple of 3, $N = 3k + 2$ say, then $N - 20 = 3(k - 6)$, so we can make up N with L and suitable S and M provided $k \geq 8$. As before, $N = 23$ cannot be made up.

But 43 is the largest of these 'impossible' numbers 3, 43, 23, so the largest impossible number of all is 43 and Clare received 86 free sweets.

92-5. BOXES OF COXES

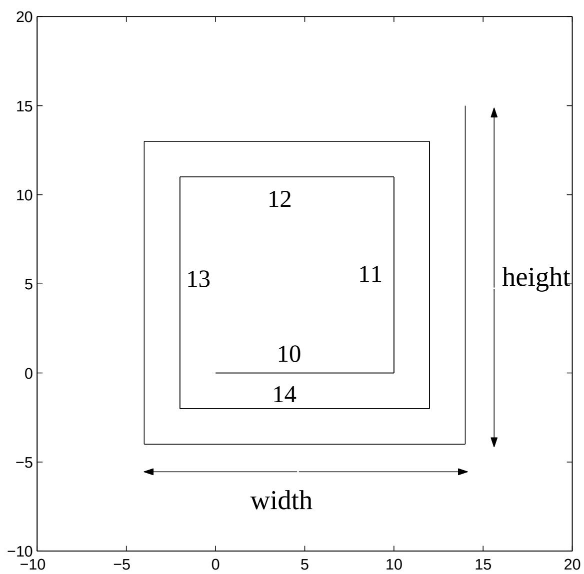
For the 4 cm apples, we get 15 rows of 15 each and 15 layers high giving $15^3 = 3375$ apples, and likewise for the 5 cm apples we get $12^3 = 1728$ apples. The total number of apples is 5103. Assuming the apples all have the same density, so that their weight is proportional to the total volume, and using the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere of radius r,
Total volume of 4 cm diameter apples is $15^3 \times \frac{4}{3} \times \pi \times (\frac{4}{2})^3$
Total volume of 5 cm diameter apples is $12^3 \times \frac{4}{3} \times \pi \times (\frac{5}{2})^3$
which come to the *same*. So the boxes are equally heavy.

92-6. SQUIRALS

From the table on the next page, after 5 steps the squiral is 60 cm long and after 10 steps it is 145 cm long. The pattern is $\frac{L}{19+n} = \frac{1}{2}n$, or $L = \frac{1}{2}n(19 + n)$.

step (n)	total length (L)	$19+n$	$\frac{L}{19+n}$
1	10	20	$\frac{1}{2}$
2	21	21	1
3	33	22	$1\frac{1}{2}$
4	46	23	2
5	60	24	$2\frac{1}{2}$
6	75	25	3
7	91	26	$3\frac{1}{2}$
8	108	27	4
9	126	28	$4\frac{1}{2}$
10	145	29	5

[If you know about sums of 'arithmetic progressions' $a+(a+d)+(a+2d)+\ldots(a+(n-1)d)$, which equals $\frac{1}{2}n(2a+(n-1)d)$, then this is the special case $a=10, d=1$.]



Referring to the figure, there is an obvious idea of the *width* and *height* of the spiral. Let's see how this varies with n:

n	3	4	5	6	7	8	9	...	n odd	n even
width	12	12	14	14	16	16	18	...	$n+9$	$n+8$
height	11	13	13	15	15	17	17	...	$n+8$	$n+9$

So the width first reaches 100, at $n=91$, when the height will be 99. We cannot go one more step because the height will become 101. So there are 91 steps inside the 1 metre by 1 metre square and this gives a total length of 5005 cm.

Senior Challenge 1993

93-1. YAPPIE FAMILIES

(i) Since there were 25 + 19 + 10 = 54 pets, and no more than 2 pets in any one house, there were pets in at least 27 houses. Given that 5 of the 32 houses had no pets, there must be pets in *exactly* 27 houses. It follows that there are no families with more than 3 children.

(ii) 25 houses have cats, thus the remaining 2 must have a rabbit and a dog.

93-2. TRUTH TABLE

The clue in the title is to tabulate the statements. There are several ways to proceed, by assuming one of the statements is true or false and using logical deduction.

	J	S	M	R	D
1st		S			J
2nd	R	M		R	
3rd	J		M		
4th				D	D
5th			S		

From the table, since David is placed 4th by two people, assume this is in fact true. Then Rachel cannot be 2nd, so Jackie is 3rd, Sue 5th, Mike 2nd, and Rachel 1st. This is consistent; on the other hand if David is *not* 4th then Jackie is 1st and Rachel 2nd; thus Sue is not 1st so Mike is 2nd. This is impossible: Rachel and Mike cannot both be 2nd. So the solution found is the only one.

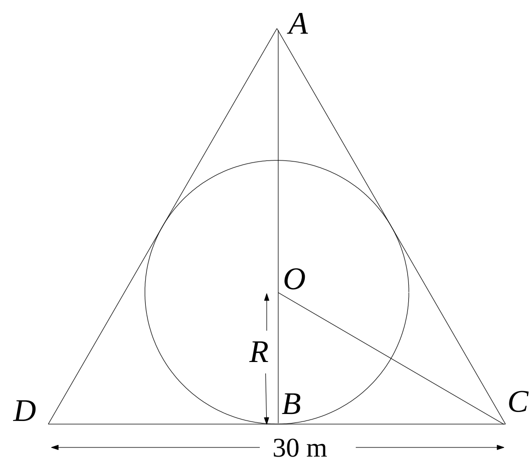
93-3. PHYSICAL DIFFERENCES

Given that 0 is one of the numbers and the largest difference is 16, this means that 16 must be present in both lists. Notice that $1+15=16, 4+12=16, 9+7=16$ where 1, 4, 7, 9, 12, 15 all come from the given differences. So let us try putting one from each pair into a list:

0, 1, 4, 9, 16;   0, 7, 12, 15, 16

It is straightforward to check that both of these work.

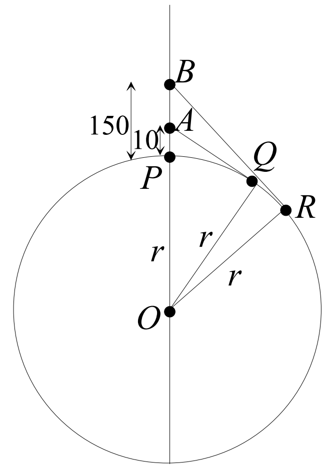
93-4. POINT TO POINT



Trigonometry is certainly the best approach here, but the following solution just uses Pythagoras's theorem. Angle ABC in the figure is a right angle and $BC = \frac{1}{2}DC = 15$ so by Pythagoras $AC^2 = AB^2 + BC^2$: $30^2 = AB^2 + 15^2$ giving $AB = 15\sqrt{3}$. From symmetry $OC = OA = AB - R = 15\sqrt{3} - R$. Using Pythagoras on triangle OBC we get $OC^2 = OB^2 + BC^2$, giving $(15\sqrt{3} - R)^2 = R^2 + 15^2$. Multiplying out gives $15^2 \times 3 - 30\sqrt{3} = 15^2$ (the R^2 cancels!) and solving we get $R = 15/\sqrt{3}$ which is about 8.7m. The area of grazing is $\pi R^2 = \pi \times 15^2/3$ sq m. On the other hand the area of the field is $\frac{1}{2}AB \times DC = 15 \times 15\sqrt{3}$ sq m and the ratio is $\pi/(3\sqrt{3})$. Multiplying by 100 to get the percentage gives 60.5%.

93-5. SETTING TIME

Again we'll give a solution which uses only Pythagoras's theorem, though trigonometry is also a possible method. Given the circumference $C = 2\pi r = 40,000$km gives $r = 20,000/\pi = 6366$km approximately. From the figure (where the heights of A and B are obviously exaggerated!) we have, by Pythagoras, $AQ^2 = OA^2 - OQ^2 = (OP + AP)^2 - r^2$, so that $AQ^2 = (r + \frac{1}{100})^2 - r^2 = \frac{r}{50} + \frac{1}{10000}$, giving $AQ = 11.28$km approximately.



Similarly $BR^2 = (r + \frac{15}{100})^2 - r^2 = \frac{3r}{10} + \frac{225}{10000}$, giving $BR = 43.70$km approximately. Now in actual fact B, A, Q and R are very nearly in a straight line, with B and A nearly coincident, and we shall get a good approximation to the arc QR by using $BR - AQ = 32.42$km. In one day the earth spins through 40,000km so it takes $(24 \times 60 \times 60)/40000$ seconds to spin 1km. The time delay of sunset is therefore this number of seconds times $32.42 = 70.03$ seconds, or about 1 minute 10 seconds.

[Here is a way of visualising what is happening. Take a sphere and mark a point (Blackpool). Draw two circles with this as centre, representing the horizons from 10m and 150m above sea level. The great circle on the sphere which separates the dark half (night) from the light half (day) turns at a steady rate, making one complete cycle in 24 hours. (The effect of the earth moving round the sun is small enough to be neglected here; it is only the 'diurnal' (daily) rotation of the earth that matters.) We want to measure the time lapse between the great circle touching one of the horizon circles and touching the other horizon circle. This is what the above calculation achieves.]

93-6. TIME WASTING

The clock has to lose 3 minutes short of a multiple of 12 hours to get back the 'lost' 3 minutes. This number of minutes must be exactly divisible by 7. Some experimentation shows that the smallest solution is 47hrs 57mins = 2877mins which divided by 7 gives 411. So it was 411 hours later that Bill visited Tom again. This is 17 days and 3 hours so it was 9pm on a Monday evening.

[In fact $60 \times 12 \times k$ is 3 more than a multiple of 7 precisely when $k - 4$ is a multiple of 7. This is because $720k - 3 = 7(107k - 1) - (k - 4)$. So for the problem we can take $k = 4$ (as in the solution) or $k = 11$, but this already makes the time interval more than a month as $(720 \times 11 - 3)/7 = 1131$ which is more than the number of hours in a month.]

Senior Challenge 1994

94-1. PRESSING PROBLEM

Assuming that 51 issues are published each year means that after 20 years, that is at the end of 2011, the number published would be $20 \times 51 = 1020$. However there are two conditions: (i) if Christmas Day falls on a Thursday or Friday the issue is lost, but (ii) if Christmas day falls on a Saturday then 31st December is a Friday so the year has 53 Fridays and an issue is gained!

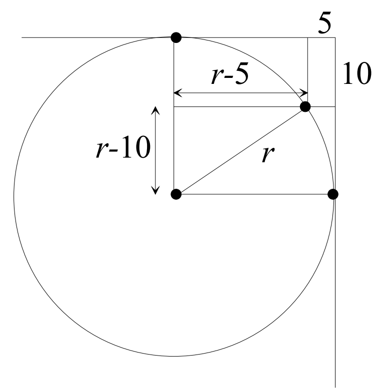
Christmas Day falls on a Thursday or Friday in 1992, 1997, 1998, 2003, 2008 and 2009 so six issues are lost. Christmas Day falls on a Saturday in 1993, 1999, 2004 and 2010 so four issues are gained. There is altogether a loss of 2 issues. Hence, at the end of 2011, there will be 1018 issues, and since 30th December is the last Friday, 18 weeks before this is 26 August 2011 for issue number 1000.

The year 2012 is a normal 51 issue year so 1069 will be the last issue in 2012, and 4th January 2013 is the date of the 1070th issue, celebrating the journal's 21st birthday.

94-2. STEP SEQUENCE

Let $N(k)$ denote the number of ways of climbing k steps. The k^{th} step can be reached from either of the two steps below, so we have $N(k) = N(k-1)+N(k-2)$. Starting from $N(1) = 1, N(2) = 2$ we can now work out the number of ways for all larger numbers of steps. Thus for 3 steps there are $N(3) = 2 + 1 = 3$ ways, for 4 steps there are $N(4) = 3 + 2 = 5$ ways, and so on. We keep adding the previous two answers. (This is an example of a *Fibonacci sequence.*) The numbers for 1, 2, 3, ... steps work out as
1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 133, 377
where the last number is $N(13)$. Since this is more than the number of days in a year Ian can indeed use a different way each day. (Maybe you can devise a systematic plan for going through all the different ways so that he never gets confused as to which he has done so far!)

94-3. CORNER TABLE



From the figure, using Pythagoras's theorem, $(r-5)^2 + (r-10)^2 = r^2$. Expanding and collecting terms we get $r^2 - 30r + 125 = 0$, that is $(r-25)(r-5) = 0$. But $r = 5$ is too small since this makes $r - 10$ negative, so $r = 25$ and the diameter is 50cm.

94-4. CLUTCHING AT STRAWS

The key thing to remember here is that in any triangle the sum of two sides must always be greater than the third side.

For 2 and 4 cm sides, the only possibilities are $(2, 3, 4), (2, 4, 4)$ and $(2, 4, 5)$.

If the largest side is 5 cm then there are these three and 19 others:
$(1, 1, 1), (1, 2, 2), (1, 3, 3), (1, 4, 4), (1, 5, 5), (2, 2, 2), (2, 2, 3), (2, 3, 3), (2, 5, 5),$
$(3, 3, 3), (3, 3, 4), (3, 3, 5), (3, 4, 4), (3, 4, 5), (3, 5, 5), (4, 4, 4), (4, 4, 5), (4, 5, 5), (5, 5, 5).$

94-5. ON YER BIKE

Without Tom, one girl would cycle to the middle (5 miles), leave the bike and walk the rest. The other girl would walk halfway, pick up the bike and cycle the other half. The time taken is: 5 miles at 12 mph = 25 min; 5 miles at 2 mph = 2hr 30min. Total time 2hr 55min.

On his own, Tom would jog the whole 10 miles in 2hr 30min at 4mph.

So the first girl can cycle past the middle and leave the bike. Tom can pick it up, cycle *back* the same distance before the middle, leave the bike and jog the rest. The second girl can pick up the bike and cycle the rest of the way. The girls gain time and Tom loses time. Now Tom cycles four times as fast as he jogs so for every minute he cycles back he needs five minutes to make up the distance. If Tom cycles for three minutes back, he puts 15 minutes on his time and finishes in 2 hr 45min. The girls' pedalling to walking ratio is 6 to 1 so for every minute the girls keep riding they gain five minutes. If each girl cycles for two minutes longer they gain 10 minutes and finish in 2hr 45min. Since the ratio of Tom's pedalling rate to the girls' is 4 to 3 the three minutes Tom is on his bike gives the girls four minutes longer, or two minutes each, so they can all manage to finish in 2hr 45min.

By algebra, let d be the distance in miles the first girl goes past the middle. For either girl the time is $\frac{5+d}{12}+\frac{5-d}{2}$ and for Tom it is $\frac{2d}{16}+\frac{10+2d}{4}$. If these are equal, to t say, then rearranging we get $12t = 35 - 5d$, $16t = 40 + 10d$, and solving we get $t = \frac{11}{4} =$ 2hr 45min.

94-6. COVER UP

Since rows (or columns) of 5 and 4 tiles are used, the problem is to find what values of the whole number n can be expressed in the form $n = 5r + 4s, r \geq 0, s \geq 0$. Clearly when n is a multiple of 5 there is no problem (take $s = 0$).

If n is 4 more than a multiple of 5, say $n = 5k + 4$, then take $r = k, s = 1$.
If $n = 5k + 3 = 5(k-1) + 8$ then take $r = k-1, s = 2$, which works so long as $k \geq 1$. But $k = 0$ gives $n = 3$ and as noted in the question this is an impossible case.
If $n = 5k + 2 = 5(k-2) + 12$ then take $r = k-2$ and $s = 3$ which works for $k \geq 2$. For $k = 0$ we get $n = 2$, also noted in the question, and for $k = 1$ we get $n = 7$, which cannot be done.
If $n = 5k + 1 = 5(k-3) + 16$ then take $r = k-3, s = 4$ for $k \geq 3$. The exceptional cases here are $n = 1, 6, 11$. The first two are noted in the question.

So the other impossible cases are 7 and 11.

Senior Challenge 1995

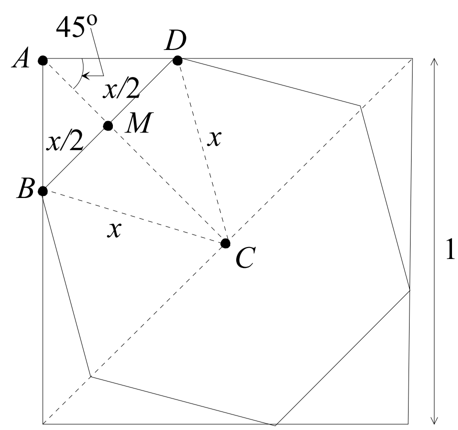
95-1. EXPIRY DATE

The moving parts will last for 2000/8 = 250 working days. The electronic components have essentially an 11 hour working day: 8 running hours, 2 hours lost on switch-on and 1 on switch-off, so the electronic components last 2500/11=227 working days and 3 hours. On the last working day the computer loses 2 hours on switch-on so works for 1 hour. Finally 227 days and 1 hour converts to 45 weeks, 2 days and 1 hour, i.e. Wednesday 15th November 1995, at 10 a.m.

Running overnight the moving parts go first. Monday 9 a.m. to Friday 5 p.m. is 104 hours which gives 2000/104=19 weeks and 24 hours, that is Tuesday 16th May 1995 at 9 a.m.

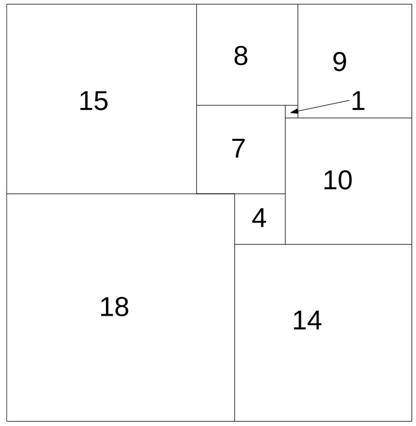
95-2. BEST SIX

In the figure, the hexagon is placed symmetrically so as to be as large as possible within the square. Now BCD is an equilateral triangle with side x, the unknown side of the hexagon. Also AC bisects BD at right-angles. Using Pythagoras's theorem, $MC = \sqrt{x^2 - (x/2)^2} = \frac{1}{2}x\sqrt{3}$. But $\frac{1}{2}\sqrt{2} = AC = AM + MC = \frac{1}{2}x + \frac{1}{2}x\sqrt{3}$. Solving this equation for x gives $x = \sqrt{2}/(1+\sqrt{3})$. Finally the area of triangle BCD is $\frac{1}{2}x \times MC = \frac{1}{4}x^2\sqrt{3}$. Substituting for x and multiplying by 6 to give the total area of the hexagon gives $3\sqrt{3}/(4+2\sqrt{3}) = 0.696$ sq m approximately.



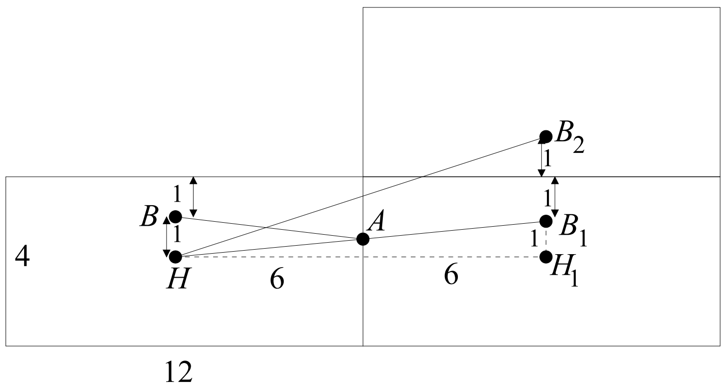
95-3. SQUARING THE RECTANGLE

See the diagram on the next page. Length = 33, width = 32.



95-4. LET THERE BE LIGHT

In the figure, B is for bulb and H for head. The actual path of the light when it bounces once off the wall is BAH, but by reflecting the room in the right-hand wall as shown we can use



instead the path B_1H which has the same length. Remember that light reflects off a wall so that the angle of incidence equals the angle of reflection. Using Pythagoras, the length of B_1H is $\sqrt{12^2+1^2}=\sqrt{145}=12.042$ approximately.

For the path which bounces off the ceiling and the wall we reflect the room twice as shown and measure the length of the path B_2H, which is $\sqrt{12^2+3^2}=\sqrt{153}=12.359$ approximately.

95-5. GREEK GODS

Let A_n denote the number of Alphites alive at the end of year n and B_n denote the number of Betons alive at the end of year n. Then from the given information,

$$A_0=4,\ A_1=2\times 4=8,\ A_2=5\times A_1=4\times 10,\ A_3=2\times A_2=8\times 10,\ A_4=5\times A_3=4\times 10^2,$$

and so on; in general

$$A_{2n}=4\times 10^n,\ A_{2n+1}=8\times 10^n.$$

Similarly

$$B_0=10^2,\ B_1=10^2,\ B_2=8\times 10^2,\ B_3=8\times 10^2,\ B_4=8^2\times 10^2,$$

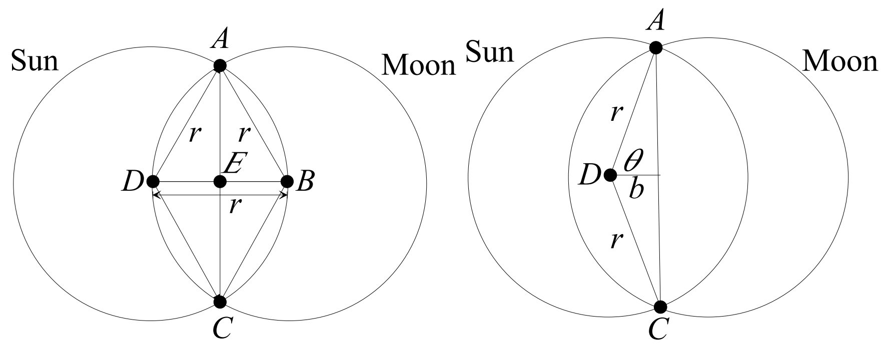
and so on; in general

$$B_{2n} = 8^n \times 10^2,\ B_{2n+1} = 8^n \times 10^2.$$

So going from an odd to an even year, A multiplies by 5 and B by 8, so A won't overtake B there. Thus it will be at the end of an odd year when first $A > B$, that is $8 \times 10^n > 8^n \times 10^2$, at the end of year $2n+1$. This is the same as $(\frac{5}{4})^n > \frac{25}{2}$ and by trial, or using logarithms if you know about them, you will find $n = 12$ is the first value, which makes it the end of the 25th year.

95-6. IN THE DARK

The left-hand figure shows the sun and moon one-quarter of the way through the 4 minute period. Let us write r for the radii of the disks. The distance between the centres of the disks is $2r$ as the eclipse begins and 0 after 2 minutes, at totality when the moon's disk exactly covers the sun's. After 1 minute, as shown, the centres B, D are r apart and therefore each circular rim passes through the centre of the other disk. The overlap is four times the area of the sector



bounded by the radii DA, DB and the circular arc AB, minus four times the area of the triangle DAE. The angle ADB is 60 degrees or $\frac{\pi}{3}$ radians, which makes the sector of area $\frac{1}{2}r^2\frac{\pi}{3} = \frac{1}{6}\pi r^2$. [The area of the sector of a circle of radius r bounded by two radii making an angle α is $\frac{1}{2}r^2\alpha$.] In fact it's pretty clear that this is correct in our case since the sector is one-sixth of the total area of the circle. The area of the triangle DAE is $\frac{1}{2}\frac{r}{2}\frac{r\sqrt{3}}{2}$ since the height AE is $\frac{r\sqrt{3}}{2}$ by Pythagoras's theorem. Putting this together the area of overlap is a percentage of the total area πr^2 given by

$$\frac{4\left(\frac{\pi r^2}{6} - \frac{1}{2}\frac{r}{2}\frac{r\sqrt{3}}{2}\right)}{\pi r^2} \times 100,$$

which comes to about 39.1%. The visible part is therefore about 60.9%.

Refer now to the right-hand figure, which represents the situation when the area of overlap is half the area of the sun's disk. The centre of the sun's disk is still called D here and the distance apart of the centres is $2b$ and $b = r\cos\theta$ where θ is so far unknown. The area of overlap is twice the area of the sector bounded by the radii DA, DC and the circular arc of the sun's disk from A to C, minus twice the area of the triangle ADC. This is to be one-half of the area of the sun's disk, that is $\frac{1}{2}\pi r^2$. Writing this out gives

$$\frac{1}{2}\pi r^2 = 2\left(\frac{1}{2}r^2(2\theta) - \frac{1}{2}r^2\sin(2\theta)\right).$$

(Here we use two formulae: (i) the area of a circular sector is $\frac{1}{2}r^2$ times the angle of the sector in radians, here 2θ; (ii) the area of the triangle ADC is $\frac{1}{2}r^2$ times the sine of the angle between the arms DA, DC.) When it is simplified this comes to

$$\pi = 4\theta - 2\sin(2\theta),$$

This is not an equation which can be solved directly, so we need to make a trial guess and improve it. Substituting in some values to the right-hand side, trying to get it equal to $\pi = 3.14159...$ gives

θ	1	1.2	1.1	1.15	1.16	1.155
right-hand side	2.1814	3.4491	2.7830	3.1086	3.1755	3.1420

The last is not at all bad so we can conclude that θ is about 1.155 radians which is 66.2° (so somewhat greater than the 60° at the one-quarter stage as in the first part of the question). Finally this makes $b = r\cos\theta = .404r$ approximately, so the distance between the centres of the disks is $2b = .808r$ approximately. The centre of the sun's disk has therefore moved a distance of $(2 - .808)r$ since the eclipse started, and recalling that the centre takes 2 minutes to move a distance of $2r$ we can conclude that the time in seconds to move to the half-visible position is

$$\frac{(2 - .808)r}{2r} \times 120 \text{ seconds},$$

which comes to 71.5 seconds.

Senior Challenge 1996

96-1. PALINDROMIC NUMBERS

$68 \to 154 \to 605 \to 1111$ so this takes 3 reversals.
$79 \to 176 \to 847 \to 1595 \to 7546 \to 14003 \to 44044$ so this takes 6 reversals.
89 is particularly slow in forming a palindrome; in fact the 24th number in the sequence is 8813200023188.

It is conjectured that every number eventually yields a palindrome, though 196 has not yet produced one! Some more information is given in [7, p.62].

96-2. GOOD FRIENDS

The table sets out the statements, with the people who make statements down the side and the statements they make along the horizontal rows.

	M	R	E	K	L	N
M	6		2			
R	1	5				
E		3	4			
K				3		6
L				3	2	
N					1	4

If K$\neq$ 3 then from the K and L rows we deduce N = 6, L = 2, but then from the N row we deduce N = 4, which is a contradiction. Hence K = 3.
We now deduce E = 4 from the E row, L = 1 from the N row, R = 5 from the R row and M = 6 from the M row. Finally N = 2. So the order from 1 to 6 is Liz, Neal, Kelly, Emily, Ross, Martin.

96-3. JOURNEY'S END

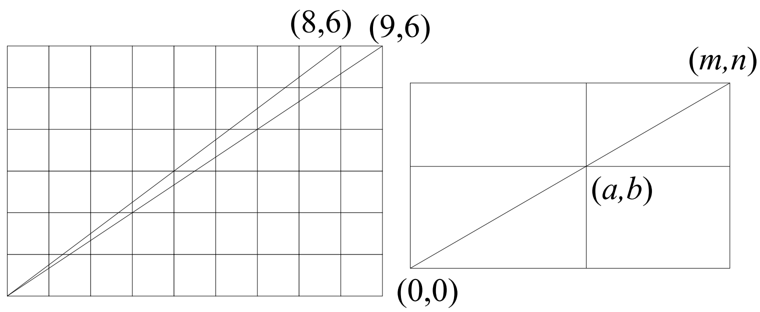
Let b be the number who board at Lime Street, which is station number 1. Then the number who board at station n is b/n so the total number boarding during the journey is is $b(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6})$. Next, if $2a$ alight at the first station after Lime Street then na alight at station n, so $7a$ alight at Euston. So $7a = 343$, giving $a = 49$. Also the total number who alight during the journey before Euston is $a(1+2+3+4+5+6)$. Since there are 343 people who arrive at Euston we have

$$b\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right) - 49(2+3+4+5+6) = 343.$$

Rearranging this gives $b = 540$.

96-4. CORNER TO CORNER

Consider an $m \times n$ rectangle of square tiles, all 1 metre square. We need to count the number of corners of interior squares which the diagonal path passes through. If it passes through no such corners then the number of tiles traversed is $m+n-1$. If the diagonal passes through c corners other than the bottom left corner (but including the top right corner) then the number of tiles traversed is $m+n-c$. So we need to express c in terms of m and n. From the right-hand figure,

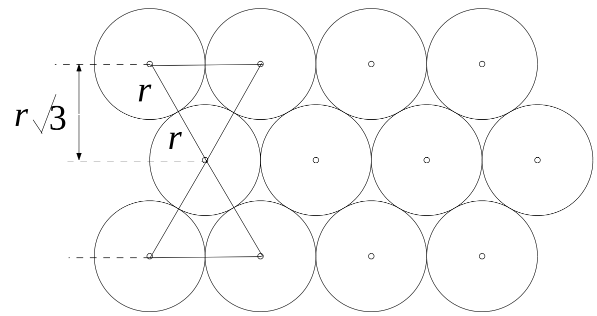


taking the origin at bottom left of the rectangle, the corner (a, b) of a square tile, where a and b are whole numbers, lies on the diagonal precisely when $m/n = a/b$. Thus a/b can be any of the ways of writing the fraction m/n where $a \le m$ and $b \le n$. Reducing m/n to its lowest terms gives say m_0/n_0, for example, $8/6 = 4/3$ or $9/6 = 3/2$, illustrated in the left hand figure. We need to divide m and n by their greatest common divisor, d say, to reduce to lowest terms, that is $m = dm_0, n = dn_0$ ($d = 2$ or 3 in the examples given). The other ways of writing m/n are $2m_0/2n_0,\ 3m_0/3n_0, \ldots, dm_0/dn_0$, so there are d of them altogether. (For $m = 9, c = 6$ these fractions are $3/2 = 6/4 = 9/6$.) This shows that c, the number of corners which the diagonal passes through, is given by $c = d = \text{gcd of } m$ and n. Thus the number of tiles traversed is $m+n-c = m+n-d$ where d is the greatest common divisor of m and n.

In the examples of the question, $m = 6, n = 8$ (clearly the same as $m = 8, n = 6$) gives an answer $6+8-2 = 12$; $m = 1, n = 6$ gives $1+6-1 = 6$ and $m = 6, n = 7$ gives $6+7-1 = 12$.

96-5. PIE SERIES

The figure shows that in the 'alternating' packing of disks of radius $r = \frac{5}{2}$, where the triangles which are drawn are all equilateral of side $2r = 5$, the vertical distance between the centres of disks in consecutive rows is $r\sqrt{3}$. (This follows from Pythagoras's theorem applied to an equilateral triangle and one of its altitudes, or from trigonometry, using $\sin 60° = \sqrt{3}/2$.) Packing disks of diameter 5 into a 60×28 rectangle in this way we can fit $60/5 = 12$ disks along one side, and we shall get n rows altogether where n is the largest whole number with $r+(n-1)r\sqrt{3}+r \le 28$. Putting $r = \frac{5}{2}$ and rearranging this gives $n-1 \le 46/(5\sqrt{3}) = 5.3$ approximately, so $n = 6$ is the biggest value, i.e. 6 rows will fit in this fashion. This makes $3 \times 12 + 3 \times 11 = 69$ disks, making 34 mince pies and one disk left over.



Using the alternative method of identical rows we get still 12 along the 60 cm side of the rectangle and only 5 rows, since each row now takes up 5 cm of 'vertical' direction, making $12 \times 5 = 60$ disks or 30 mince pies. Clearly the alternating method is much better.

On the other hand if Joanna continually makes the pastry into a new ball, rolls it out to 3mm thick and makes more circular cuts, then she can keep making disks until the volume of pastry does not permit having a circle of radius 5 cm. This means that she can make

$$\frac{60 \times 28}{\pi(\frac{5}{2})^2} = 85.56$$

disks, that is 85 disks, which are good for 42 mince pies and one disk left over.

Finally, with Wendy's 2mm thick pastry the initial area will be $60 \times 28 \times 3/2 = 2520$ sq cm, and the same calculation as before gives

$$\frac{2520}{\pi(\frac{5}{2})^2} = 128.34$$

disks, that is 128 disks, which are good for 64 mince pies.

96-6. EVEN BREAK

It helps greatly in the solution of this problem if you know the formula

$$1 + 2 + 3 + 4 + \ldots + n = \tfrac{1}{2}n(n+1).$$

(The classic and beautiful proof of this[9], is to reverse the series and add:

$$\begin{array}{ccccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \ldots & + & (n-2) & + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & (n-2) & + & \ldots & + & 3 & + & 2 & + & 1 \end{array}$$

Adding gives $2S = (n+1) + (n+1) + \ldots + (n+1)$ where there are n terms added, so $2S = n(n+1)$.)

So the triangular arrangement of billiard balls with n rows has $\frac{1}{2}n(n+1)$ balls. What we are looking for, then, is whole numbers x (of red rows) and y (the total number of rows) where

$$\tfrac{1}{2}x(x+1) = \tfrac{1}{2}(\tfrac{1}{2}y(y+1)), \quad \text{that is,} \quad 2x(x+1) = y(y+1). \tag{7}$$

An equation such as this, containing two unknowns x and y which are *whole numbers*, is called a *Diophantine* equation, after Diophantus, a Greek mathematician from the classical era, who studied equations of this kind (but not so hard as this one!).

[9] According to legend this was (re)discovered by the great German mathematician C.F.Gauss at the age of 10.

From equation (7) we get $y^2 + y - 2x(x+1) = 0$ and solving this quadratic equation for y,

$$y = \frac{-1 \pm \sqrt{1+8x(x+1)}}{2},$$

where we have to take the + sign in front of the square root to make y positive. It follows that $z = \sqrt{1+8x(x+1)}$ has to be a whole number, and indeed *odd* to make y a whole number. Looking at the ratios of successive values of x in the given table (the second column contains the values of x, the number of red rows of balls), they are $14/2 = 7, 84/14 = 6, 492/84 = 5.86, 2870/492 = 5.83, 16730/2870 = 5.82926829$ approximately, so we expect the next entry will be an even number slightly less than $16730 \times 5.82926829 = 97523$ approximately. So we start looking for such an x which makes $z = \sqrt{1+8x(x+1)}$ a whole number:

x	z
97522	275835.28
97520	275829.63
97518	275823.97
97516	275818.31
97514	275812.66
97512	275807

Aha! This gives $x = 97512, y = 138903$.

Using more advanced techniques, it can be shown[10] that the *general* solution for x and y can be calculated from the 'recurrence relations'

$$x_1 = 0,\ y_1 = 0;\quad x_{k+1} = 3x_k + 2y_k + 2,\ y_{k+1} = 4x_k + 3y_k + 3.$$

For example, $x_2 = 2,\ y_2 = 3;\ x_3 = 3\times 2 + 2\times 3 + 2 = 14,\ y_3 = 4\times 2 + 3\times 3 + 3 = 20$, and so on.

As for the ratio of x to y, equation (7) gives

$$2\frac{x}{y} = \frac{y+1}{x+1}.$$

When x and y get very large, the right hand side is very nearly $\frac{y}{x}$. So if $\frac{x}{y}$ gets close to r say then we expect $2r = 1/r$ and solving this gives $r^2 = \frac{1}{2}$ or $r = 1/\sqrt{2}$. You can also take the ratios of values of x and y in the table and see that they are getting close to 0.707... which you *might* recognise as about $1/\sqrt{2}$.

Senior Challenge 1997

97-1. BEE LINE

Tracing the family tree backwards from Dennis Drone, the number of males in any generation of his ancestors equals the total number of bees two generations closer to him, and the number of females in any generation equals the total number of bees one generation closer to him. So the total number of bees in generation n back from Dennis equals the sum of the numbers in generations $n-1$ and $n-2$. (Except for $n = 1$ when there is one ancestor.) This means that the generations go like the Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = F_1 = 1$. The numbers in the generations are $1, 2, 3, 5, 8, 13, 21, 34$ with a total of 87 for the first eight generations back from Dennis.

[10]The editors are grateful to Professor E.V.Flynn for this result.

97-2. WHIZZY LIZZY

The most complete way to do this is to use some algebra, but many entrants wrote down convincing arguments about the cancellation of middle digits which were also acceptable.

A number with digits abc equals $100a + 10b + c$ so we are looking at

$$10(100a + 10b + c) - (100b + 10c + a) = 999a,$$

since the contributions of b and c cancel.

For the second part, the same formula shows that if 37 is a factor of $100a + 10b + c$ then, as 37 is also a factor of 999 ($= 27 \times 37$), it follows that 37 is a factor of the remaining term $100b + 10c + a$.

Several entrants noticed that the same holds in the last trick if we replace 37 by any other factor of 999, e.g. 27.

97-3. IN A FLAP

The volume of the given box is (area of base) $\times$ (height) which is $16 \times 1 = 16$ cubic centimetres. With squares of side x cut out the volume is $x(6-2x)^2$ and the difference in volumes is

$$16 - x(6-2x)^2 = 16 - 36x + 24x^2 - 4x^3.$$

From here there are various approaches. You could draw up tables of values, or sketch a graph. (A more advanced method uses calculus.) You can also factorize the last expression as

$$4(1-x)^2(4-x),$$

which is certainly ≥ 0 as the square is ≥ 0 and $x \leq 4$ (in fact only $x \leq 3$ makes sense for the cardboard model). Thus the volume is always ≤ 16 and the original value $x = 1$ gives the maximum volume.

97-4. HAVING IT TAPED

There are basically two approaches. One calculates the average radius of a 'circle' of tape (actually part of a spiral) as $\frac{1}{2}(1.1 + 2.2) = 1.65$ cm., so that the $4.76 \times 30 \times 60 = 8568$ cm. of tape are arranged in about

$$\frac{8568}{2 \times \pi \times 1.65} = 826$$

layers. The thickness is then $\frac{2.2-1.1}{826} = 0.0013$ cm.

The second approach uses area. The area of the edge of the tape is the difference between the areas of circles radius 2.2 cm. and 1.1 cm., which is $\pi(2.2^2 - 1.1^2) = 11.404$ sq. cm. Thinking of the tape as unrolled and looked at edge on as a long thin rectangle, the area of this rectangle is then 11.404 sq. cm. and its length is 8568 cm. as above so the thickness of the tape is about $\frac{11.404}{8568} = 0.0013$ cm.

97-5. TRUTH TO TELL

The only possible sequence of scores after the first seven weeks is 6, 2, 7, 3, 8, 4 and after that you run into problems because the next score has to be > 8 and < 1, an impossibility. So six further weeks is the maximum for truth telling.

The general argument can be seen, as suggested, from a diagram where you use the rules to put an arrow from a bigger score to a smaller one. If you try to extend the scores to week 14 (that is, 7 weeks after the initial 7), you get a chain of arrows which *closes up*, which is impossible since scores go down steadily along the chain. Using $[n]$ to indicate the score on week n, you get the impossible chain

$$[7] > [14] > [12] > [10] > [8] > [6] > [13] > [11] > [9] > [7].$$

Each step uses one of the rules that going back 7 weeks increases the score and going back 2 weeks decreases it. [The original version of this question comes from the Leningrad Mathematical Olympiad (it still seems to be called that), which is an *oral* mathematics championship in which competitors are confronted with problems and try to solve them at sight on a blackboard. No hints were given and only the general case was considered.]

97-6. ROUGH RIFFLES

All the pairs in the first case contain one black and one red card, and *all* the sets of four cards in the second case contain one card from each suit. Explaining this is not so easy, however. The first 'trick' works if you just *cut* the pack with cards of different colours on top of the two parts, but to make the two 'tricks' uniform they were both given in the same way.

Let us take the second 'trick', which is slightly more complicated. Write a_1, a_2, a_3, a_4 for the four suits in the given order. From an initial arrangement (top of pack to bottom)

$$a_1, a_2, a_3, a_4, a_1, a_2, a_3, a_4, \ldots, a_1, a_2, a_3, a_4$$

we get, after dealing about half the pack to the table, two parts of the form

$$\begin{array}{ccc} a_k, \ldots, a_1, a_4, a_3, a_2, a_1, & \ldots, & a_4, a_3, a_2, a_1, \\ a_{k+1}, \ldots, a_4, a_1, a_2, a_3, a_4, & \ldots, & a_1, a_2, a_3, a_4 \end{array}$$

where k can be 0, 1, 2, 3 or 4, and if $k = 0$ then the first segment of the first part is absent while if $k = 4$ then the first segment of the second part is absent. In the first part each segment has *decreasing* suffix, and in the second part each segment has *increasing* suffix.

Now rough riffle the two parts together. The top 4 cards of the shuffled pack consist of

the first	0	1	2	3	4	cards from the first part, and
respectively the first	4	3	2	1	0	cards from the second part.

In all cases the top 4 cards form a complete set of suits. Now remove these cards—it's easiest to think of removing them from the two parts before shuffling. The situation is still the same, but with possibly a different value of k. Repeating the argument we find that all subsequent sets of 4 cards have one from each suit.

Of course, the same thing works if we arrange the original pack in groups of any number besides 2 or 4, e.g. in groups of 13 with the face values 1, 2 up to K in each group.

Senior Challenge 1998

98-1. HOME MOVIE

They arrive back 10 minutes earlier than usual, so that they must have met 5 minutes in car-speed time from the station, i.e. they meet at 4.55 p.m., since the wife timed it to get to the station at 5 p.m. Therefore, the man was walking for 55 minutes.

98-2. STONE AGES

Let the ages be a, b, c, d, e respectively. Then from the information given $(ab)(bc)(cd)(de)(ea) = 36 \times 9 \times 8 \times 24 \times 12$, so that $(abcde)^2 = 6^2 \times 3^2 \times 8^2 \times 6^2$, giving: $abcde = 6 \times 3 \times 8 \times 6$.

So, $36 \times 8 \times e = (ab)(cd)e = 6 \times 3 \times 8 \times 6$. Hence, $e = 3$. Substituting back: $a = 4, b = 9, c = 1, d = 8$.

98-3. STEPTOE AND SON

Let the man walk x km and ride $(32-x)$ km to halfway. Then the time taken is: $x/3+(32-x)/8$ hours. The son rides x km and walks $(32-x)$ km, so his time is: $x/8+(32-x)/4$ hours. Since they meet at half way, both times are the same:

$$\begin{aligned} x/3 + (32-x)/8 &= x/8 + (32-x)/4 \\ 8x + 3(32-x) &= 3x + 6(32-x) \\ 5x + 96 &= 192 - 3x \\ 8x &= 96 \\ x &= 12. \end{aligned}$$

Time taken is: $12/3 + (32-12)/8 = 6.5$ hours. Total time for journey is: $6.5 + .5 + 6.5 = 13.5$ hours. They arrive at 7.30 p.m.

98-4. IN THE CLEAR

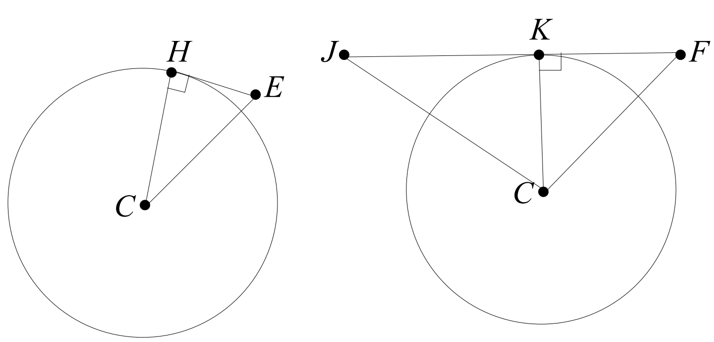
For each couple, with the woman buying W items and the man buying M items, we have $W^2 - M^2 = 63 = 3^2 \cdot 7$. Thus $(W-M)(W+M) = 63 = 63 \cdot 1 = 21 \cdot 3 = 9 \cdot 7$. Clearly, the only possibilities for $W+M$ & $W-M$ are: 63 & 1, 21 & 3, 9 & 7, so that the only possibilities for W & M are: 32 & 31, 12 & 9, 8 & 1. Louise bought 23 items more than Quentin, giving that Louise bought 32 and Quentin 9. Melanie bought 11 items more than Peter, so that Melanie bought 12 and Peter 1. Nicola and Richard must have bought the remaining possible number of items: namely Nicola bought 8 and Richard 31. So, we finally get the couples: Louise & Richard, Melanie & Quentin, Nicola & Peter. Total number of objects: $32 + 31 + 12 + 9 + 8 + 1 = 93$.

98-5. DIAMOND CUT DIAMOND

Pair the numbers as follows: (1 and 7), (2 and 6), (3 and 9), (4 and 8), (5 and 10). Note that any member of a pair can always be played immediately after its companion. Player two will always win, with the strategy that, after each first-player-move, player two plays the other member of the pair. Note that other correct styles of answer are possible, such as a tree enumerating all possibilities.

98-6. PIG AHOY

Draw the Earth as a circle, and draw a line from the Earth's centre C to Jonah's eyes E, a line from Jonah's eyes to the horizon (this line will be tangent to the circle at H), and from that



point on the horizon to the earth's centre. Thus CHE is a right angled triangle with hypotenuse $CE = 6.3 \times 10^6 + 2$, and the side CH is 6.3×10^6. By Pythagoras's theorem, the side EH is then: $\sqrt{(6.3 \times 10^6 + 2)^2 - (6.3 \times 10^6)^2}$ which is 5020 m (to the nearest metre), or 5.02 km. So, the horizon appears 5.02 km away from Jonah in his dinghy. (Note that in the figure the heights above sea level are greatly exaggerated relative to the radius of the earth!)

In the second situation, we draw a line from Jake's eyes J to the top of Jonah's flag F, which is tangent to the Earth since Jake can *just* see the top of the flag. Thus, there are two right angled triangles. The distance JK is $\sqrt{JC^2 - KC^2} = \sqrt{(6.3 \times 10^6 + 15)^2 - (6.3 \times 10^6)^2}$ which is approximately 13747.73 m. The distance KF is similarly $\sqrt{(6.3 \times 10^6 + 3)^2 - (6.3 \times 10^6)^2}$ which is approximately 6148.17 m. So, the total distance in a straight line from Jake's eyes to the top of Jonah's flag is: 19895.90 m, or approximately 19.9 km.

Senior Challenge 1999

99-1. DIVINE ASSEMBLY

$528 = r(17 + 2s)$, where r is the number of rows, and $2s$ is the number of seats outside the aisles in any given row. The factorisation of 528 is: $528 = 2^4 \cdot 3 \cdot 11$, so clearly 33 is the only odd factor of 528 equal to or greater than 17. It follows that $17 + 2s = 33$ and the number of rows is: $r = 528/33 = 16$.

99-2. FOREVER EASTER

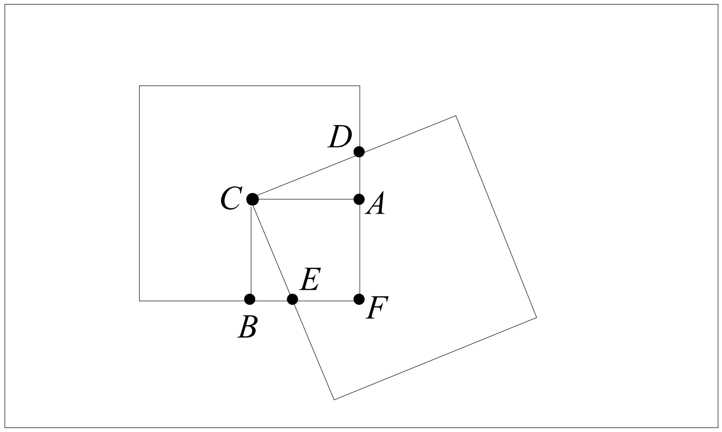
Let A, B, C be the number of Ants, Bunnies, Chickens he buys, respectively. Then $5A + 160B + 82C = 30000$ and $A + B + C = 365$, so that $5A + 160B + 82(365 - A - B) = 30000$. This is the same as: $-77A + 78B = 70$. The clever thing to do here is to notice that $-77 + 78 = 1$, and so multiplying both sides by 70 gives $-77 \cdot 70 + 78 \cdot 70 = 70$. That is, we can take $A = B = 70$. We can also add 78 to A and add 77 to B to get another solution $A = 148, B = 147$. These are the only solutions to $-77A + 78B = 70$ for which A, B are nonnegative and $A + B$ does not exceed 365, the total number of items. (Note that the general solution for A,B is $A = 70 + 78k, B = 70 + 77k$ for any integer k; clearly $k = 0, 1$ are the only allowable values which make A, B nonnegative and $A + B$ not exceeding 365.) When $A = B = 70$, we have $C = 365 - 70 - 70 = 225$. When $A = 148, B = 147$, we have $C = 365 - 148 - 147 = 70$. In summary, the only possibilities are: $A = 70, B = 70, C = 225$ and $A = 148, B = 147, C = 70$. We are also told that C is the smallest, so the answer must be the second option: $A = 148, B = 147, C = 70$.

99-3. HECTOR'S HOUSE

Let x be the distance walked before the bus comes into view/earshot; since his walking speed is the same uphill and downhill, we know that x will be the same whichever stop he chooses. Once the bus comes into view/earshot, it will take $90/15 + 8 = 14$ seconds before it departs the first stop, and so Hector's time to spare will be: $14 - (90 - x)/3$. Once the bus comes into view/earshot, it will take $90/15 + 8 + 90/15 + 270/15 + 8 = 46$ seconds before it departs the second stop, and so Hector's time to spare will be: $46 - (270 - x)/5$. Setting these equal to each other gives $x = 60$, and substituting this into either expression gives 4 seconds as the time to spare. If the left hand stop is chosen, then the time taken is: $60/2 + 210/5 = 72$ seconds, plus the 4 seconds to spare gives a total of 76 seconds until the second bus goes round the left hand corner. Adding this to the 18 mins 44 seconds in the house gives exactly 20 mins between the buses.

99-4. GROTTY PAINTING

As in the figure, drop perpendiculars from the centre C of the square to two sides, meeting the sides at A and B. The triangles CAD, CBE are congruent ($CA = CB$, right-angles at A

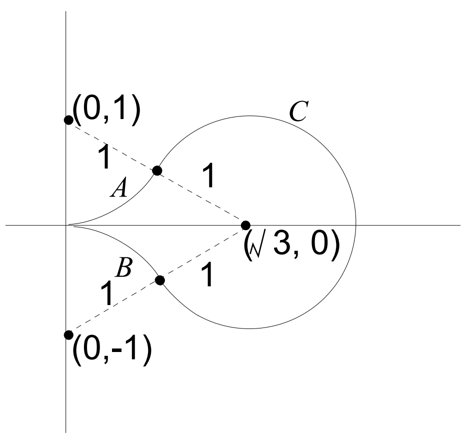


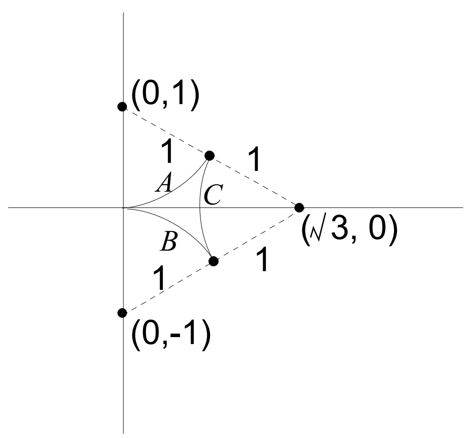
and B, and equal angles BCE, ACD since these are both equal to the angle the titled square is turned relative to the other square). So these two triangles have the same area, and the 'red' area of the painting is always equal to the area of the square $ACBF$, that is one-quarter of the area of the larger squares in the picture. The red area is always the same!

99-5. ABOUT TURN

The first given route is a straight line from $(0, 0)$ to $(2, 0)$ [length 2], followed by an arc from $(2, 0)$ to $(1, 1)$ anticlockwise around angle $3\pi/2$ of the circle centred at $(2, 1)$ and radius 1 [length $3\pi/2$], and finally an arc from $(1, 1)$ to $(0, 0)$ clockwise around angle $\pi/2$ of the circle centred $(0, 1)$ and radius 1 [length $\pi/2$]. Total length of the route is therefore: $2 + 3\pi/2 + \pi/2 = 2 + 2\pi$.

The second given route is an arc from $(0, 0)$ to $(0, 2)$ anticlockwise around angle π of the circle centre $(0, 1)$ and radius 1 [length π], followed by an arc from $(0, 2)$ to $(0, -2)$ anticlockwise around angle π of the circle centre $(0, 0)$ and radius 2 [length 2π], followed by an arc from $(0, -2)$ to $(0, 0)$ anticlockwise around angle π of the circle centre $(0, -1)$ and radius 1 [length π]. Total length of the route is therefore: $\pi + 2\pi + \pi = 4\pi$.





The construction for the shortest routes is as follows. Draw circles A and B, each of radius 1, with centres $(0, 1), (0, -1)$, respectively. Then A and B touch each other at $(0, 0)$. Draw a third circle C, also of radius 1, with centre on the x-axis, which touches circles A, B. In the figure, only the arcs of the circles A, B, C are drawn which are needed for Fanny's route.

The lines joining the centres of the three circles make an equilateral triangle, with sides of length 2. The x-axis cuts this into two right angled triangles, each with hypotenuse 2 and height 1. So, the base has length $\sqrt{(2^2-1^2)} = \sqrt{3}$. So, the centre of circle C is $(\sqrt{3}, 0)$. Since the equilateral triangle formed by the lines joining the centres of A, B, C has three angles of 60 degrees (or $\pi/3$ radians), we see that the small arcs on each of the circles between the touching points are each of length one-sixth of the circumference, that is $\pi/3$ units, since each circle has radius 1. When Fanny can only drive forwards, the shortest route is first the short $\pi/3$ arc around A from $(0,0)$ to the touching point of A and C (which is $(\sqrt{3}/2, 1/2)$, although there is no need to compute this), followed by the long $5\pi/3$ arc around C to the touching point of B and C (which is $(\sqrt{3}/2, -1/2)$, although there is no need to compute this), followed by the short $\pi/3$ arc around B back to $(0,0)$. This has total length: $\pi/3 + 5\pi/3 + \pi/3 = 7\pi/3$.

When Fanny can also reverse, she can drive forwards along the short $\pi/3$ arc around A from $(0,0)$ to the touching point of A and C, then reverse along the short $\pi/3$ arc around C to the touching point of B and C, then drive forward along the short $\pi/3$ arc around B back to $(0,0)$. This has total length: $\pi/3 + \pi/3 + \pi/3 = \pi$. [Extra for experts: this last can be seen to be the smallest possible, since the circular arcs are the only way to change direction, so that arcs adding to a half circle must be included somewhere in the route in order to reverse direction. But our last route above has length the same as a half unit circle, and so is the shortest possible.]

99-6. BLOCKBUSTERS

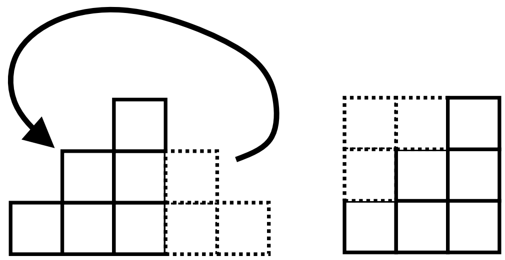
In the two by two case, the X must be in a corner (by symmetry it does not matter which corner); let's say it is the top left hand corner. The second player has a winning strategy as follows. Whatever the first player does, the second player is handed either $\begin{matrix}\text{X} & \text{O}\end{matrix}$ or $\begin{matrix}\text{X}\\ \text{O}\end{matrix}$. The second player can win by eating an O and giving the first player the X. Conclusion: in the two by two case, the second player can always force a win.

In the three by three case, suppose that the X is either in a corner or in the centre. Imagine a line L drawn through the X square along a diagonal of the chocolate bar. The second player has a winning strategy as follows. At every move, whatever line the first player has just broken along, the second player breaks along the reflection of that line in L (this means that the bar is always square-shaped when it is handed back to the first player). Eventually, the first player must be left with the X. [Of course, the same argument could also have been used for the two by two case]. It remains to consider when the X is in the middle of a side, say: $\begin{matrix}\text{O} & \text{X} & \text{O}\\ \text{O} & \text{O} & \text{O}\\ \text{O} & \text{O} & \text{O}\end{matrix}$. Then the first player can force a win as follows. First, the first player eats the bottom two rows, and hands $\begin{matrix}\text{O} & \text{X} & \text{O}\end{matrix}$ to the second player; from now on, the game proceeds exactly as for that described in the question. Conclusion: in the three by three case, the second player can force a win when the X is in a corner or the centre; otherwise, the first player can force a win.

In the four by four case, imagine that the X is either in a corner or is one of the four inner squares. Then we can draw a line L through X along a diagonal of the chocolate bar, and repeat the second-player-win strategy described above. It only remains to consider when the X is a non-corner outer square, say: $\begin{matrix}\text{O} & \text{X} & \text{O} & \text{O}\\ \text{O} & \text{O} & \text{O} & \text{O}\\ \text{O} & \text{O} & \text{O} & \text{O}\\ \text{O} & \text{O} & \text{O} & \text{O}\end{matrix}$. The second player then has a winning strategy as follows. For the first move, whatever line the first player breaks along, the second player breaks along the reflection of that line in the bottom-left-to-top-right diagonal. This means that, after one move each, the first player will receive one of: $\begin{matrix} \mathrm{X} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} \end{matrix}$, $\begin{matrix} \mathrm{O} & \mathrm{X} \\ \mathrm{O} & \mathrm{O} \end{matrix}$ or $\mathrm{O} \quad \mathrm{X} \quad \mathrm{O}$, all of which we have already seen to be second player wins. Conclusion: in the four by four case, the second player can always force a win.

Senior Challenge 2000

00-1. CUT AND COVER



00-2. SPINNING FREDDY

Let d be the distance Fred rides. Then

$$\frac{1}{3}d + \frac{1}{7}d + 1 = \frac{1}{2}d, \quad \text{i.e. } d\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{7}\right) = 1,$$

giving $d = 42$ miles.

00-3. HIGH POWERED

$$2^3 > 7, \quad \text{so} \quad (2^3)^{1000} > 7^{1000}, \quad \text{i.e.} \quad 2^{3000} > 7^{1000}.$$

$$2135 = 5 \times 427; \quad 5978 = 14 \times 427, \quad 7^5 = 16807, \quad 2^{14} = 16384 < 7^5,$$

these being possible by hand or on an ordinary calculator, so

$$7^5 > 2^{14}, \quad \text{giving} \quad 7^{2135} = (7^5)^{427} > (2^{14})^{427} = 2^{5978}.$$

00-4. ALL IN GOOD TIME

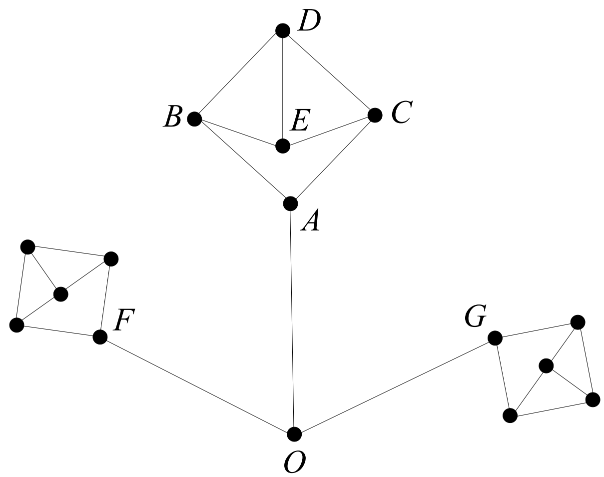
There will be $100 \times 12 = 1200$ months. There will be $365 \times 100 + 25$ days, counting the 25 leap years 2000, 2004, ..., 2096 as giving an extra day each. This is 36525 days, which is 5217 weeks and 6 days, i.e. 5217 complete weeks. [The same is true of any century, even one that does not begin with a leap year, for the number of days then goes down by 1 making 5217 weeks and 5 days.]

00-5. JOAN AND JIM

It *is* impossible. For counting 3 friends for each of the 11 people, we count each pair of friends *twice*, i.e. $33 = 2\times$ the number of pairs, which is impossible. Clearly the number of party-goers must be *even* for everyone to have exactly 3 friends.

00-6. TALL STORY With n layers there are exactly n^2 bricks, using a cut such as that in Question 1 to turn the wall (in our mind's eye!) into a perfect square. There are 5217 weeks (and 6 days) by Question 4, so we want the nearest square to 5217, which is $72^2 = 5184$. So there should be $n = 72$ layers and $2n - 1 = 143$ blocks in the bottom layer. We want to calculate the date of the Saturday which is exactly 5184 weeks after 1 January 2000. Besides these 5184 weeks there are 33 weeks and 6 days = 237 days up to and including 31 December 2099. Counting backwards from 31 December 2099, calling this 'day one', we want 'day 237'. Counting back to 1st May 2099 from 31 December gives 245 days (1 May is 'day 245'), so 'day 237' is 9 May 2099: on this Saturday the last brick will be laid.

00-7. JIM AND JOAN It *is* impossible. Consider the 5 dots (people) in one of the 'end-groups', such as A, B, C, D, E shown in the figure.



If A is paired to B or C then the remaining three among A, B, C, D, E cannot be paired. So A must be paired to the central dot O. But the same applies to the other two endgroups, which says that three dots (people), namely A, F, G all have to be paired with O. This is impossible.

Note that it *is* true that everyone has three friends, with the even number 16 of partygoers, in this question.

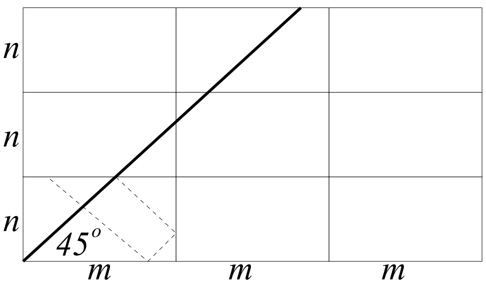
00-8. MILLENIARDS The numbers of bounces are:

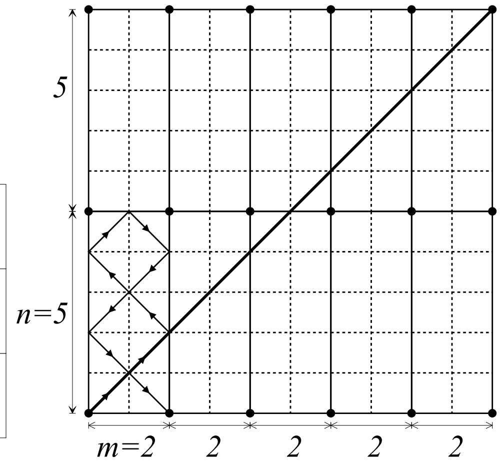
3×5: 6 bounces; 4×5: 7 bounces; 5×5: 0 bounces; 6×5: 9 bounces; 7×5: 10 bounces; 8×5: 11 bounces; 9×5: 12 bounces; 10×5: 1 bounce.

It is pretty clear that $m \times n$ and $km \times kn$ tables give the same number of bounces, for $k = 1, 2, 3, \ldots$: the whole table is just expanded by a factor of k. For instance, 1×2 and 5×10 tables will have the same number of bounces (namely 1). So we can assume from now on that m, n have no common factor–for instance 25×35 will give the same number of bounces as 5×7, where we have taken out the common factor 5.

The trick now is to repeatedly 'open out' the table by reflection. In the figure, an $m \times n$ table is opened out a few times; the dashed line is the beginning of the actual path of the ball in the bottom left table, but the thick line is this path as it appears when you repeatedly reflect the table in one of its sides. The dashed line becomes a *straight* 45° line!

Suppose we reflect as many times as is needed to make an $mn \times mn$ square. The case of 2×5 is illustrated in the figure. The diagonal 45° line, which represents the path of the ball in the 'opened out' table, passes through the top right-hand corner of this square. When m and n have no common factor the line *will not pass through any corner of a $m \times n$ rectangle before this one.* (These corners are indicated by heavy dots in the example.) This is a crucial property, and





a proof is given at the end of this question. (In the original competition we did not of course expect such formal proofs.)

The number of bounces is then the number of horizontal or vertical lines in the figure which are crossed by the diagonal line–not the dashed lines which represent the division of the rectangle into unit squares, but the solid lines which represent the edges of the table. Every crossing of an edge corresponds to a bounce. But the number of edges crossed is $m-1$ horizontal ones and $n-1$ vertical ones (1 horizontal line and 4 vertical lines in the example) before the ball ends in the top right-hand corner of the diagram, which means that it has entered a corner of the table. Thus the number of bounces is $m+n-2$.

The rule is: Take out any common factor from m and n and work out $m+n-2$ for this reduced pair of numbers. For example 25×35 would given 5×7 and then $5+7-2=10$ bounces.

From the diagram we can also read off the results (again taking out any common factor from m and n):
If n is even then the ball hits a corner of the table on the left side, i.e. top left corner.
If m is even then the ball ends on the right side, i.e. bottom right corner.
If m and n are both odd then the ball ends in the top right of the table.
(Note that m and n cannot both be even as they have no common factor!)

For 2000 bounces we need $m+n-2=2000$, that is $m+n=2002$. So we take any two numbers m and n which have no common factor and which add to 2002, and use a $m\times n$ table. We can also use a $km\times kn$ table for any such m,n and any k. I believe that there are 360 pairs (m,n) with $m+n=2002$ and m,n having no common factor, counting (m,n) and (n,m) as the same pair. That's an awful lot of ways of making 2000 bounces!

Here is a proof that when m and n have no common factor, the diagonal line does not pass through any corner at coordinates (km,ln) before $k=n,l=m$, the corner at (mn,mn). Since the line is at 45° we have $km=ln$. But this implies that ln is a multiple of m. *Since m and n have no factor at all in common, this is only possible if l is a multiple of m.* So $l=m$ is the smallest possible value, and from this and $km=ln$ we get $k=n$.

Senior Challenge 2001

01-1. PEN FRIENDS

The three sides of length b add to 12 m, so $4a=36$, giving $a=9$m. The area of each pen is $ab=36$ sq m.

01-2. SECOND THOUGHTS

This can be solved by trial and error, or by algebra. Using algebra we get $4a + 3b = 48$ and $ab = 48$. From these we deduce

$$4a + 3\frac{48}{a} = 48a, \text{ giving } 4a^2 + (3 \times 48) = 48a, \text{ i.e. } a^2 - 12a + 36 = 0,$$

after rearranging and cancelling 4 from both sides. This gives $(a-6)^2 = 0$ so $a = 6$ is the only solution. The sides are $a = 6$, $b = 48/6 = 8$. Notice that in this case, a is the shorter side.

01-3. SQUARE DEALS

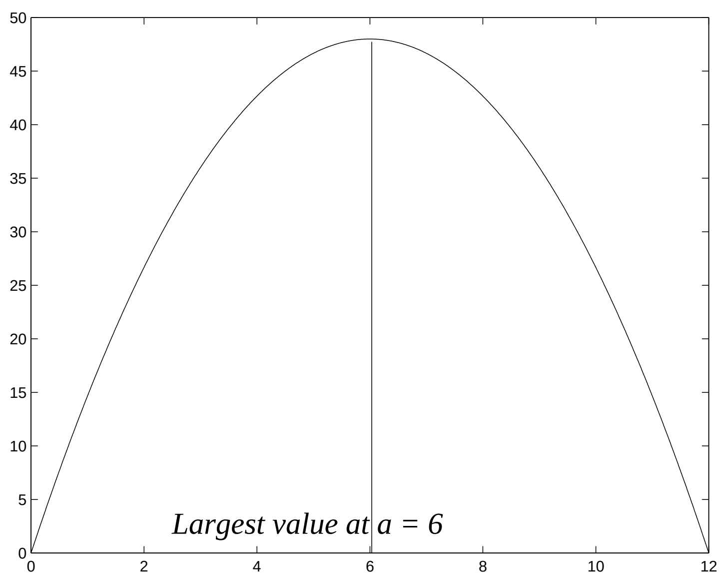
The interesting thing is that all the answers come out to be the same in the bottom line:

Chosen card	1	2	3	4	5	6	7	8	9
Place after one deal and pickup	1	1	1	2	2	2	3	3	3
Place after two deals and pickups	1	1	1	1	1	1	1	1	1

The question doesn't actually specify the order in which the columns besides the one containing the chosen card are picked up, but it's not hard to see this doesn't matter anyway. Notice the simple progression after one deal and pickup.

01-4. A PENSIVE CHAT

There are several ways to see that Farmer Chris is right. We can use a bit of algebra and a graph, or more algebra. We know that $4a + 3b = 48$ and we're interested in how big ab can be. Put $ab = A$ for Area. Then $4a + \frac{3A}{a} = 48$ which gives $A = 16a - \frac{4}{3}a^2$. The question is: how big



can $16a - \frac{4}{3}a^2$ be? We could take several values for a and plot a graph, as in the figure, or we can argue without a graph as follows:

$$48 - A = \frac{4}{3}a^2 - 16a + 48 = \frac{4}{3}(a^2 - 12a + 36) = \frac{4}{3}(a-6)^2 \geq 0,$$

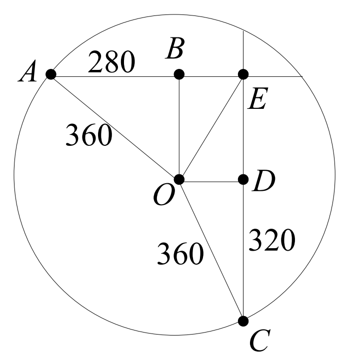
since squares are always ≥ 0. So always $48 - A \geq 0$ so $A \leq 48$ as Chris said. We found in Question 2 that $A = 48$ requires $a = 6$ and $b = 8$. This is a very clever kind of algebraic argument and we mainly expected you to take values and plot a graph or something like that.

01-5. OH FOR A MOON!

At first it seems that the best time is 19 minutes, with A running back and forth taking the others over one by one. But you can do better than that if the two slowest go over *together*. There's no point in doing that unless there is a faster person on the other side who can then come back. The best solution, which can be found by intelligent trial and error, is (indicating the people on the two sides of the bridge by separating them with a vertical bar |):
ABCD | ; CD | AB ; BCD | A ; B | ACD ; AB | CD ; | ABCD
The time used is $2 + 2 + 10 + 1 + 2 = 17$ minutes. It doesn't matter whether B comes back first or A, in either case the time is 17 minutes.

01-6. GARDEN OFFCENTRE

This question can be done entirely by Pythagoras's theorem (as you would expect since it is Pythagoras who is doing the planting).

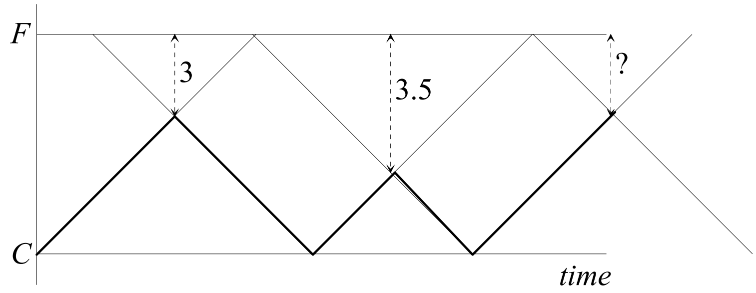


Referring to the figure, OB and OD are perpendicular to the two strings, and so B and D are the midpoints of the strings. Thus AB is half the length of one string, hence 280 cm, and CD is half the length of the other string, hence 320 cm. Using right-angled triangle ABO we find $OB^2 = 360^2 - 280^2 = 40^2(9^2 - 7^2) = 40^2 \times 32$ and using right-angled triangle CDO we find $OD^2 = 360^2 - 320^2 = 40^2(9^2 - 8^2) = 40^2 \times 17$. But notice also that $OBED$ is a rectangle, so that $BE = OD$. We now get $OE^2 = BE^2 + OB^2 = OD^2 + OB^2 = 40^2(17 + 32) = 40^2 \times 7^2 = 280^2$, so that $OE = 280$ cm.

(In general, if the strings are of length $2a, 2b$, r is the radius of the circle and c is the distance OE, then the same method as above gives $2r^2 = a^2 + b^2 + c^2$. An alternative, slightly harder version of this problem is to give a, b, and c and ask for r. Also notice that all the numbers a, b, c, r in the problem are whole numbers. Maybe you can find other values which make this true (other than just multiplying everything by the same amount). You have to be a little careful, since it's essential to have a and b both less than r.)

01-7. SOME FUN!

This question is quite well solved by means of a diagram, as shown on the next page. Here C and F stand for Charlie's house and Frankie's house and the vertical axis is distance. The thin line represents Frankie's progress, with time along the horizontal and the scale chosen so that his line is at 45° to the horizontal. The line is *straight* because Frankie moves at a constant speed, so travels the same distance in each unit of time. Likewise Charlie's progress is marked by the bold line, and, because they run at the same speed, his line is also at 45° to the horizontal. You can now see from the diagram the places where they meet and turn round; note that these make straight lines, because Charlies's and Frankie's lines are both 45° lines. You can also see the distances of 3 miles and 3.5 miles. (It is now clear that Charlie starts first, as in the figure.) Also it is clear that the diagram starts to repeat after Charlie's second visit home, so the answer to the place where they meet for the third time is 3 miles from Frankie's house.



Notice that it's really just the same thing as Charlie running back and forth straight from his house to Frankie's, passing Frankie on the way who is running back and forth from his house to Charlie's.

01-8. BIG DEALS

With a 5×5 array of cards, the position of the card after one deal and pickup (the first row of the table) takes the form
1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5
and the row under that is all 1's: the chosen card always comes to the first place after two deals and pickups.

In fact with a little thought it becomes clear that what is happening is this. Let there be r rows and c columns. Suppose we know the position p of the chosen card after a particular deal and pickup. To find the position after the *next* deal and pickup, divide p by the number of columns c and take the smallest whole number $\geq \frac{p}{c}$. Let us write this whole number as $\{\frac{p}{c}\}$. For example, with $r = 5$ and $c = 3$, the positions 1, 2, 3, 4, 5, 6, 7, 8, ... at the start become $\{\frac{1}{3}\} = 1, \{\frac{2}{3}\} = 1, \{\frac{3}{3}\} = 1, \{\frac{4}{3}\} = 2, \{\frac{5}{3}\} = 2, \{\frac{6}{3}\} = 2, \{\frac{7}{3}\} = 3, \{\frac{8}{3}\} = 3$, and so on. For the next deal and pickup we can repeat the process with these new positions 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5 to obtain 1,1,1,1,1,1,1,1,1,2,2,2,2,2,2 and one more deal and pickup will make all the numbers 1: every choice of card at the beginning will come to the first position after at most three deals and pickups.

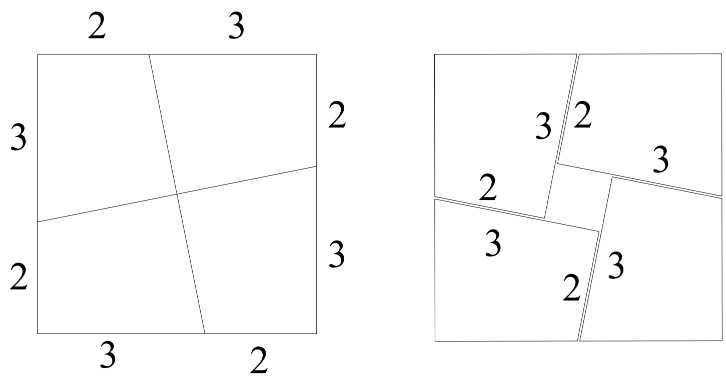
So in general, with r rows and c columns, how many deals and pickups does it take for the chosen card to be certain to be in first place? When $r = c$ it's pretty clearly always *two* deals and pickups, as in the 3×3 and 5×5 cases above. In fact this will also be true if $r < c$. The general answer is in fact $\{\frac{r}{c}\} + 1$ deals and pickups to guarantee that the chosen card is in first place.

You can make a variant of this problem by picking up the column with the chosen card *second* instead of first. In that case is it true that the chosen card will always come to the same position after a suitable number of deals and pickups?

Senior Challenge 2002

02-1. SQUARES

From the diagram, the hole is a 1×1 square.



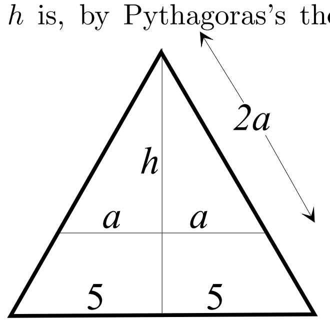
02-2. A RUSSIAN TALE

Let A, B, C be the numbers of mushrooms collected by Anton, Boris and Carla respectively. Then A is 80% of B, i.e. $A = \frac{4}{5}B$, making $B = \frac{5}{4}A = 375$. Similarly C is 120% of B, that is $C = \frac{6}{5}B$, so $C = 450$.

02-3. HELIPAD

Here is a solution which avoids the use of trigonometry.

The key observation is that if we draw a line across the triangle so as to cut off a length of $2a$, as shown, then the line cuts off another equilateral triangle (of side $2a$), so the height h is, by Pythagoras's theorem, $h = a\sqrt{3}$, and the area above the line is $a \times a\sqrt{3} = a^2\sqrt{3}$.



In particular, of course, the area of the whole triangle is $25\sqrt{3}$ sq.m. One-third of this is $25\sqrt{3}/3$ and we want to draw the first line so that the area above it equals this amount, so we want $a^2\sqrt{3} = 25\sqrt{3}/3$, that is $a = 5/\sqrt{3}$. The height h of this triangle is $a\sqrt{3} = 5$, that is we need to draw the first line a distance of 5 below the apex of the triangle.

For the second line, of length $2b$, say, we want $b^2\sqrt{3}$ equal to *two-thirds* of the total area, that is $50\sqrt{3}/3$. This gives $b = \sqrt{(50/3)}$ and the corresponding height is $b\sqrt{3} = \sqrt{50}$: the second line needs to be drawn at a distance below the apex of $\sqrt{50} = 7.07$ to 2 decimal places. (The whole height of the helipad triangle is $5\sqrt{3} = 8.66$ to 2 decimal places.)

02-4. THE PARTY'S OVER

Here is an algebraic solution; of course intelligent trial and error will also produce the answer. Let there be m married couples altogether, and s single people. If everyone said goodbye to everyone else then all the $2m + s$ people would say goodbye to $2m + s - 1$ others, and there would be $(2m + s)(2m + s - 1)$ goodbyes said. But we have to subtract the $2m$ goodbyes which are not said by husbands to wives and vice versa. So the total is

$$4m^2 - 4m + 4sm + s^2 - s = 102.$$

(Notice incidentally that this implies $4m^2 - 4m < 102$ and so $m(m - 1) < 25$ and $m \leq 5$. But in practice we would probably take $m = 0, 1, 2, 3...$ in succession and use the equation above to try to find s.) Each m actually gives a quadratic equation for s, and if the formula for solving a quadratic is not known, then trial will quickly decide whether there is a whole number solution. For example with $m = 1$ (so no married guests) we get $3s + s^2 = 102$ which by trying $s = 1, 2, 3 \ldots$ quickly shows that no whole number solution exists. In fact $m = 4$ is the first value which gives a solution, with the corresponding value of s equal to 3. So there were $m - 1 = 3$ married couples besides the hosts at the party.

02-5. NOW FOR THE WASHING UP

Of course, Gerald ended with the same number that he began with. Here some algebra makes the process much easier to explain; otherwise all one can do is to look at many cases.

Let the original figures be a, b so that the actual value of the number chosen by Gerald is $10a + b$. Then the number obtained by subtracting the digits from 9 is $10(9 - a) + (9 - b) = 99 - 10a - b$.

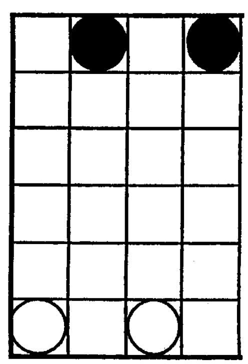
Next, the two numbers are put together. This is the same as multiplying the first number by 100 and adding the second one:

$$100(10a + b) + 99 - 10a - b = 990a + 99 + 99b.$$

Dividing by 11 gives $90a + 9 + 9b$: notice this shows that the result will always be a whole number.

Subtacting 9 gives $90a + 9b$ and dividing by 9 gives $10a + b$, which is what Gerald started with. Hey presto!

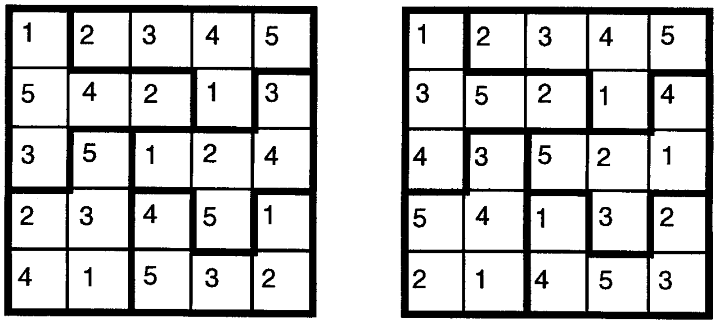
02-6. CAN SHE DO IT?



Yes, Lou can, by simply moving her left-hand counter steadily forwards, that is *downwards*, leaving for later the other one, which could not be taken off anyway by Mike, by the rules of the game. It is only after every two moves that Mike can have his counters in a position to capture Lou's. The danger moment is after Lou's third move, when the best that Mike can do is to guard her next square. But he needs two moves to shift his attack to the square that Lou is on, by which time she has slipped through.

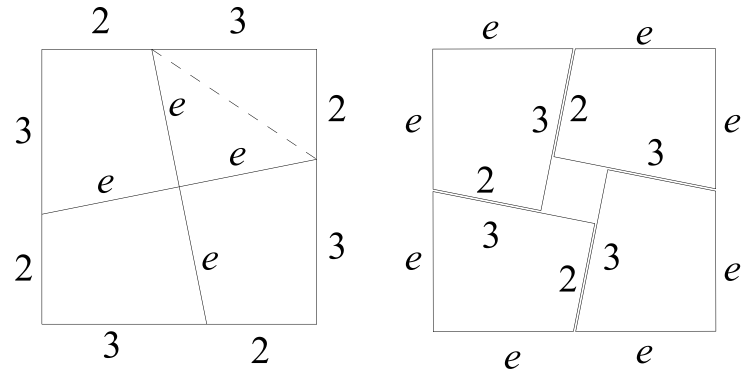
02-7. PENTAJIG

There are several solutions to this one. We have used numbers for the five colours. With the first row as shown here there is only one place for 1 in the second row, and only two possible places for 1 in the third row. Further choices have led to the two solutions shown.



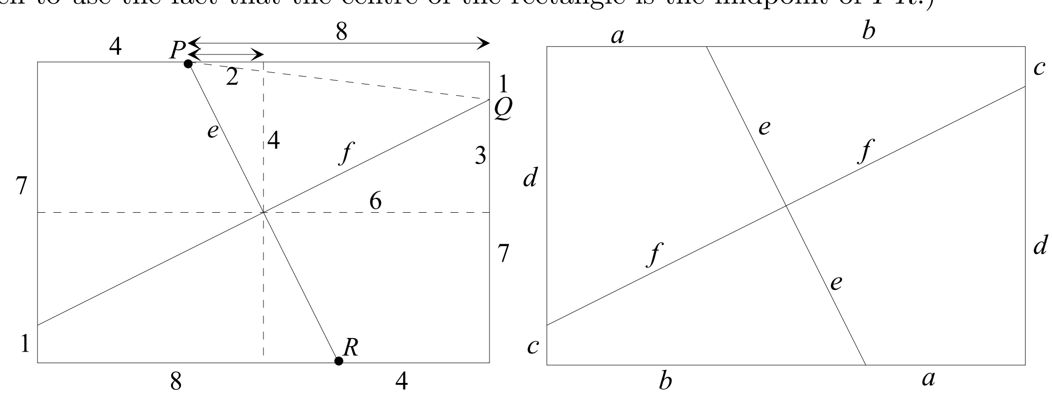
02-8. Rectangles

In this figure, the larger square has side $2e$ and using Pythagoras's theorem on two triangles which have the dashed lines as hypotenuse, we get



$e^2 + e^2 = 3^2 + 2^2$ so $e^2 = 6.5$ which makes $2e = 5.099$ to 3 decimal places. So you can see this square is only a tiny bit bigger than the original one. (You could also say that, using areas, $(2e)^2 - 1 = 25$, as the hole has area 1 and the pieces making up the original square, of area 25, have only been rearranged. This gives $(2e)^2 = 26$, giving the same answer for $2e$.)

In the figure, dashed lines have been drawn through the centre of the rectangle, parallel to the sides of the rectangle, and the lengths 2, 3, 4, 6 marked come from simple subtraction of lengths. It follows that $e^2 = 20$ and $f^2 = 45$ by Pythagoras's theorem. Thus $e^2 + f^2 = 65$. But the square of the length PQ is, also by Pythagoras's theorem, $8^2 + 1^2 = 65$. Finally using Pythagoras's theorem yet again, it follows that the angle at which the sides e, f meet is a right angle. (Another way of seeing this is to show that $PQ = QR$ using Pythagoras's theorem and then to use the fact that the centre of the rectangle is the midpoint of PR.)

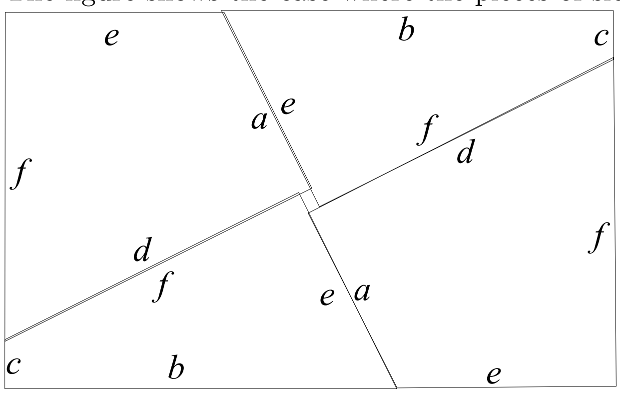


In the second figure, the lengths have been replaced by general lengths a, b, c and d, and the condition that the lines meet at right-angles works out as $a^2 + d^2 = b^2 + c^2$

For finding the different ways of reassembling the pieces, a paper or cardboard model is much recommended! In the pictures below, we use the general letters a, b, c, d; for the problem given, substitute $a = 4, b = 8, c = 1, d = 7$, in which case $e = 2\sqrt{5} = 4.47...$, $f = 3\sqrt{5} = 6.71...$ using a similar argument to the first one given above for the square. Notice, by the way, that $e/f = (c + d)/(a + b)$ in the special case. (Actually, this always holds, and is not too hard to prove using the fact that each of the four pieces is bounded by a cyclic quadrilateral!)

There are three cases: we can turn over the pieces of sides a, e, f, d; we can turn over the other two pieces, or we can avoid turning any pieces over. (If we turn them all over then this doesn't give anything different.)

The figure shows the case where the pieces of sides a, e, f, d are turned over.



The rectangular hole is here $e - a$ by $d - f$, which in the special case is 0.47 by 0.29 approximately, or exactly $2\sqrt{5} - 4$ by $7 - 3\sqrt{5}$.

When the pieces with sides b, c, f, e are turned over we get a rectangle in the middle with sides $b - e$ and $f - c$, which in the special case is 3.53 by 3.71 approximately, or $8 - 2\sqrt{5}$ by $3\sqrt{5} - 1$ exactly.

In this case, the shapes of the rectangles are the same:

$$\frac{2\sqrt{5} - 4}{7 - 3\sqrt{5}} = \frac{3\sqrt{5} - 1}{8 - 2\sqrt{5}},$$

as can be verified by cross-multiplication. It is an interesting fact that this always holds: assume that $b > e > a, d > f > c$ as in the example. Then the rectangles formed as above have the same shape, that is

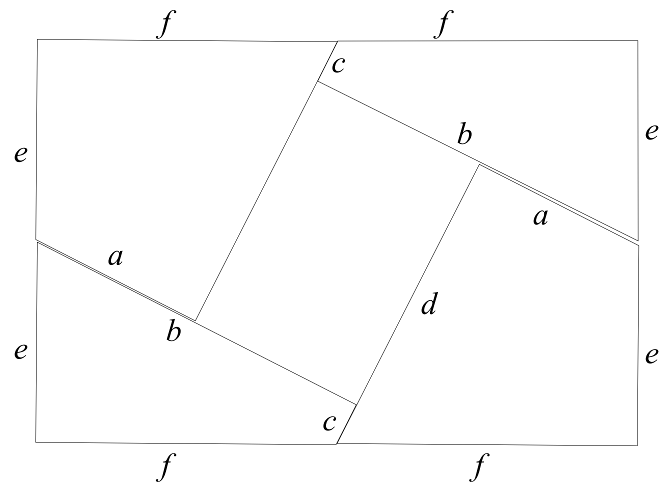
$$\frac{e - a}{d - f} = \frac{f - c}{b - e}.$$

We leave this as a nice exercise!

If we don't turn over any of the pieces then the rectangle in the middle becomes $b - a$ by $d - c$, when $b > a, d > c$ as in the example (where this rectangle is 4 by 6). We always have

$$\frac{b - a}{d - c} = \frac{c + d}{a + b}$$

since cross-multiplying this gives the known fact $b^2 - a^2 = d^2 - c^2$. So this rectangle is always the same shape as the original. This case is illustrated below.



Senior Challenge 2003

03-1. CUISENAIRE

The largest possible length is $8 + 4 + 2 + 1 = 15$ and experiment shows that all smaller lengths can in fact be made with the given rods. So the answer is 16. [In fact this is just the same as writing a number in base 2: for example $11_{10} = 1011_2$, that is 11 in base 10 is written 1011 in base 2, indicating that rods 8, 2, 1 are needed. More generally, lengths of $1, 2, 4, \ldots 2^n$ make up all lengths $< 2^{n+1}$.]

03-2 UNEXPECTED GUEST

Suppose that a fraction x of each piece is cut off; this represents a fraction $\frac{x}{6}$ of the whole pizza. Putting the six cut-off pieces together makes a fraction $6 \times \frac{x}{6} = x$ of the whole pizza. But we want this to equal $\frac{1}{7}$ so that all seven guests receive one-seventh of the pizza. So $x = \frac{1}{7}$.

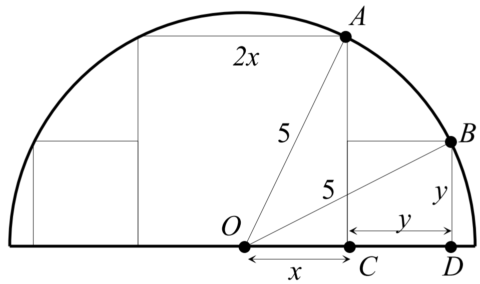
03-3 GOOD DOGS

When Fido is finished with a piece of carpet the number of pieces has increased by *three* (one piece becomes four) and when Trusty is finished with a piece the number has increased by *six* (one piece becomes seven). So however many times either dog does his or her thing, the number of pieces at the end will have the form 1 + a multiple of 3. Now $2003 = 667 \times 3 + 2$, that is *two* more than a multiple of 3. So 2003 cannot be the full tally of carpet pieces: there must be at least two more hidden away somewhere.

Note that this depends on the special numbers 4 and 7, each one more than a multiple of 3. If these are replaced by say 4 and 6 then we are asking which whole numbers can be written in the special form $1 + 3n + 5m$ for whole numbers n and m. This is also an interesting problem, and in fact every number ≥ 9 can be written in this way, so any final number of carpet pieces from 9 upwards is possible.

03-4 GARDENERS' QUESTION TIME

This is tricky but can be done entirely with Mrs Pythagoras's husband's theorem about right-angled triangles. (In fact legend has it that it was exactly to solve this problem that Mr Pythagoras invented his theorem.) The radius of the circle is 5, and using the notation of the figure $OA = OB = 5$.



Let the large square have side $2x$. By symmetry the centre of the circle must be at the midpoint of the bottom side of the larger square for this square to fit neatly inside the semicircle. Using (Mr) Pythagoras's theorem on triangle OAC gives $x^2 + (2x)^2 = 5^2$, which gives $5x^2 = 25$, that is $x = \sqrt{5}$. Using the same theorem on triangle OBD we get $(x + y)^2 + y^2 = 5^2$, which gives $x^2 + 2xy + 2y^2 = 25$ and on using $x = \sqrt{5}$ we get $y^2 + y\sqrt{5} - 10 = 0$. This can be written $(y - \sqrt{5})(y + 2\sqrt{5}) = 0$ and the only positive solution to this is $y = \sqrt{5}$. So in fact the small square has exactly half the side length of the large square. The total area of the squares is $(2x)^2 + 2y^2 = 30$, using $x^2 = y^2 = 5$. The total area of the semicircle is $\frac{1}{2}\pi \times 5^2 = 39.27$ approximately. So the area left for planting is about 9.27 square metres.

Note. By dividing the large square into four by horizontal and vertical lines and by a careful use of symmetry about the 45° diagonal line, it is possible to see that the large square has exactly twice the side of the small square (that is, $x = y$) directly.

03-5 DOING THE SPLITS

Let n and m be the two three-figure numbers. Putting n to the left of m makes the number $1000n + m$. So we are trying to find solutions of

$$1000n + m = 7nm. \tag{8}$$

The hint is intended to suggest that you try $n = m$ since the left side is then $1001n$, making the equation $1001n = 7n^2$ or $1001 = 7n$. As 1001 is divisible by 7 this leads to $n = m = 143$.

But to get bonus marks you should check that there are no other solutions. We are looking for 3-figure numbers so n and m lie between 100 and 999. Now (8) can be rewritten as $m = n(7m - 1000)$, showing that m is an exact multiple of n, but because of the restriction on the size of m and n it also shows that $7m - 1000$ must be one of the numbers $1, 2, 3, \ldots, 9$. We have only to examine these values one by one. Taking $7m - 1000 = 1$ gives the answer already obtained, $m = n = 143$. The other values apart from 8 do not give whole numbers for m, and 8 gives the slightly phony answer $m = \frac{1008}{7} = 144$: phony because it leads to $n = 18$, not a three-figure number. This is the 'solution' $018144 = 7 \times 18 \times 144$.

For the second part we are trying to solve

$$1000n + m = nm.$$

As before rewriting this gives $m = n(m - 1000)$. But $m \leq 999$ so the right hand side is negative, showing this to be an impossibility. Of course you can analyse other cases such as $1000n + m = 2nm$ in the same way.

03-6 RUNNING MATES

We need to use the formula 'time taken = distance travelled divided by speed'. Let d be the distance between the houses, let c be Chris's speed and let a be Alex's speed. When they first meet,

$$\frac{d - 600}{a} = \frac{600}{c}, \text{ that is } \frac{a}{c} = \frac{d - 600}{600}.$$

When they meet for the second time,

$$\frac{2d-400}{a} = \frac{d+400}{c}, \text{ that is } \frac{a}{c} = \frac{2d-400}{d+400}.$$

We now have two expressions for $\frac{a}{c}$ and equating them gives

$$\frac{d-600}{600} = \frac{2d-400}{d+400}, \text{ that is } d^2 - 1400d = 0$$

after cross-multiplying and simplifying. The only solution apart from $d = 0$ (which hardly seems to be practical!) is $d = 1400$ metres.

Notice that we cannot find the actual speeds of the runners from this, only the ratio of their speeds, $\frac{a}{c}$, which comes to $\frac{4}{3}$.

A number of entrants noticed a second solution: 800m. This is the solution when, at the second meeting, the runners are travelling *in the same direction*. It requires that $\frac{a}{c} = \frac{1}{3}$.

03-7 SQUARE BASHING

The key to this is really symmetry. With a board which has both dimensions even or both odd, the first player should paint a square whose centre is at the centre of the board. After that, every move by the second player should be answered by the first player by painting a square which is symmetrically placed to the second player's, by a rotation through 180° about the centre of the board. The first player is always assured of a legal move and so must finish painting the board.

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